

Quadratic Factors

Our next example is one step more complicated. The degree of the numerator is still less than the degree of the denominator, but the denominator Q will have a quadratic factor:

$$\frac{x^2}{(x-1)(x^2+1)}.$$

Notice that there's nothing more you can do to factor the denominator.¹

The set-up for this example looks like:

$$\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

The difference is that the numerator corresponding to the factor (x^2+1) is linear, not constant; in general it will be a polynomial with degree one less than the degree of the denominator.

We can still use the cover-up method to solve for A :

$$\begin{aligned} \frac{x^2}{\cancel{(x-1)}(x^2+1)} &= \frac{A}{\cancel{x-1}} + \frac{\cancel{Bx+C}}{x^2+1} \\ \frac{1^2}{1^2+1} &= A \\ A &= \frac{1}{2} \end{aligned}$$

It turns out that the best way to solve for B and C is the “slow way”; to clear the denominators completely by multiplying both sides by $(x-1)(x^2+1)$.

$$\begin{aligned} \frac{x^2}{(x-1)(x^2+1)}((x-1)(x^2+1)) &= \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1} \right) ((x-1)(x^2+1)) \\ x^2 &= A(x^2+1) + (Bx+C)(x-1) \\ x^2 &= Ax^2 + A + Bx^2 + Cx - Bx - C \\ x^2 &= (A+B)x^2 + (C-B)x + (A-C) \end{aligned}$$

The coefficient of x^2 on the left hand side is 1. On the right hand side, the coefficient of x^2 is $A+B$. We know that $A = \frac{1}{2}$, so we get:

$$\begin{aligned} 1 &= A+B \\ 1 &= \frac{1}{2} + B \\ B &= \frac{1}{2} \end{aligned}$$

¹If you use complex numbers you can further factor the denominator and again use the cover-up method; in fact, this method was originally intended to be used with complex numbers. This technique was first used by Heaviside in his work with Laplace transforms and inversion of differential equations, but it also turns out to be very useful for integration.

The constant term on the left hand side is 0. On the right hand side the constant term is $A - C = \frac{1}{2} - C$. We conclude that:

$$C = \frac{1}{2}.$$

Question: Why didn't you use the coefficient of x^1 ?

Answer: It turns out that I didn't need it. From experience, I know that the highest and lowest degree terms of the product will be the easiest terms to work with. I avoid those when possible because they usually involve more arithmetic.

Question: Could you just set $x = 0$?

Answer: Absolutely. That's equivalent to equating the coefficients of the constant term. "Plugging in numbers" is an alternate method of solving these equations, but you still want to avoid plugging in the same number twice.

Question: What if we had $x^3 + 1$ in place of $x^2 + 1$?

Answer: If there's an $x^3 + 1$ in the denominator we haven't fully factored the denominator; we're back to step 1. This method of simplifying rational expressions doesn't work if we don't factor the denominator in the first step. Steps 1 and 2 would become:

$$\begin{aligned} \frac{x^2}{(x-1)(x^3+1)} &= \frac{x^2}{(x-1)(x+1)(x^2-x+1)} \\ &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \end{aligned}$$

Question: If the degree of the numerator equals the degree of the denominator, none of this works?

Answer: It definitely doesn't work. We're going to have to do something totally different to handle that case.

Remember that the point of all this work was to make it possible to compute an integral:

$$\begin{aligned} \int \frac{x^2}{(x-1)(x^2+1)} dx &= \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1} \right) dx \\ &= \int \left(\frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1} \right) dx \\ &= \int \frac{\frac{1}{2}}{x-1} dx + \int \frac{\frac{1}{2}x}{x^2+1} dx + \int \frac{\frac{1}{2}}{x^2+1} dx \end{aligned}$$

We've split the integral of the rational expression into three simpler integrals. The first is easy to solve by substituting $u = x - 1$, the second can be solved by

“advanced guessing” or by substituting $v = x^2 + 1$. The last one can be solved by trigonometric substitution or by remembering the antiderivative of $\frac{1}{x^2 + 1}$.

$$\int \frac{x^2}{(x-1)(x^2+1)} dx = \frac{1}{2} \ln|x-1| + \frac{1}{4} \ln(x^2+1) + \frac{1}{2} \tan^{-1} x + c.$$

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