

Hi. Welcome back to recitation. In class, one of the things you've talked about recently was computing surface areas of solids of rotation. So I have a nice problem relating to that here. So the circle with center $(R, 0)$ and radius little r -- so this is center big R , 0 , and radius little r , which is less than big R -- is rotated around the y -axis. And the question is, what's the surface area of the resulting solid?

So we have here the circle. Its center is at the point big R , 0 , and its radius is little r , so this is the equation of that circle. And we're going to rotate this circle around this axis. So we're going to spin it around. And what you're going to get is a donut, or what mathematicians call a torus. So here's a little schematic of it here, with one dotted little cross section corresponding to this circle.

So the question is, what is the surface area of this torus? So why don't you pause the video, take a few minutes to work this problem out yourself, come back, and we can work it out together.

So when we solved surface area problems in class, we took the curve that we were going to rotate, and we cut it into lots of little pieces with length ds . So let me draw that, just very quickly, up here. So we had, you took whatever your curve was, and you cut it into lots of these little segments. And then for each segment, you rotated it around an axis. And so if the segment has this little length ds , a little piece of arc length-- so in our case, we're going to rotate it around the y -axis, so the length, the area that this thing traces out as it spins around, is going to be this little piece of area, dA , which is equal to $2\pi x ds$.

Now, this is a little bit different than most of the examples Professor Jerison did in class, because here we're rotating around the y -axis, not around the x -axis. So if you rotated around the x -axis, what you would get is $2\pi y ds$. And here we get $2\pi x ds$. The idea of the x and the y in this formula, this x , it's just telling you what the radius is between your little segment and the axis around which you're rotating it. So here this is, you know, $2\pi x$ is the circumference of this circle that it traces out, and ds is its length, because it's giving you a little ribbon as it goes around.

So we have this formula, dA equals $2\pi x ds$. And so in order to get the surface area, what we do, is we have to integrate this over an appropriate region. So in order to do that, we first need, you know, all the variables in the integrand to be the same, so we need to write everything in terms of x , or everything in terms of y , or everything in terms of some variable that we can integrate against.

So in our particular case with this torus, I think we can take advantage of a little bit of symmetry here, which is that this, you know, torus is top-bottom symmetric, right? As the top half of the circle goes around, it traces out one surface. As the bottom half of the circle goes around, it traces out another surface. But those two surfaces have

exactly the same area. They're just mirror images of each other, because the circle is symmetric.

So we can just consider the problem of spinning the top half of the circle around. And so for the top half of the circle, we can write down an equation for y in terms of x , and so then we can integrate-- you know, that sets up a nice integral with respect to x .

So in order to do this, well we're going to need two things. So we're going to need to know what ds is. And so you had a couple of different formulas for this in class. So you had ds -- so one easy mnemonic that I like is to write ds equals the square root of dx squared plus dy squared. So for me, this always-- I can remember this, because it's just the Pythagorean theorem, right? So you have a little dx horizontal distance, and a little dy vertical distance, and so the ds is just a hypotenuse of that triangle. So that's how I remember this.

And so then you also have the equivalent formula if you factor out a dx from here. You know, divide through by dx and multiply outside by dx . You can write this as the square root of 1 plus dy by dx squared times, outside the square root, dx .

So this is our little piece ds . And OK, we have an x in the formula already. So ds -- so we now have ds with a dx here. So the only thing we have left, is we need to figure out what dy/dx is in order to put all this into the formula, in order to integrate. So OK. So dy/dx . So OK, so that means I need y in terms of x . Now let-- OK. So if we're focusing only on the top half of this torus, that means y -- well, we can solve this equation for y , but when we take the square root, we're only taking the positive square root, because we're only looking at the top half of the torus, and then we'll just double at the end to account for the bottom half as well.

So this is y equals the square root of-- so you solve, you subtract x minus r squared from both sides. So this is little r squared minus x minus big R squared.

So that's y . And now you can differentiate to get dy by dx . All right. So now we have to do our chain rule right. So we've got a $1/2$ power. So this is $1/2$ times-- well, it's going to be the inside to the minus $1/2$. So this is over the square root of r squared minus x minus R quantity squared, and now I need to multiply by the derivative of the inside, which is minus 2 times x minus R . OK. And the 2 's cancel out a little bit, so we can rewrite this as minus x minus R divided by the square root of little r squared minus x minus R squared.

OK. So that's dy/dx . The square root-- that's a little ugly, but it's OK, because we're about to square it out again. So all right. So we've got dy/dx . So now we go back, now we can compute ds , and then with ds , we can go even further back, and we can compute dA . And once we've got dA , to get a , we just integrate dA over the appropriate bounds. So-- which we haven't figured out yet, by the way. We will have to talk about that at the very, you know, in a minute.

OK. So we take this, so we have dy/dx . So from dy/dx , we get ds is equal to-- it's the square root of, well, let's use this formula. It's the square root of 1 plus, now, dy/dx squared. OK. So I put this in and I square it. So I get x minus R squared on top. You know, you square the minus sign, gives you 1 . And on the bottom, we square the square root, so we get little r squared minus x minus R squared, OK, dx .

Great. Good. So now we add these two things together, right? I mean, well we want to-- I should say-- we want to simplify this to a usable form. And so to put it in a usable form, well we're going to manipulate it until it looks nice, or as nice as we could hope for. And in this case, right. So we can, there's an obvious simplification, or simplifying step, which is we can add this 1 into the fraction. OK. And so this is little r squared minus x minus big R squared, over little r squared minus x minus big R squared, and so when you add it to x minus big R squared over little r squared minus x minus big R squared, this part cancels. And so you're left with-- OK. And now I'm just going to pass the square root through right away, so this is little r over the square root of little r squared minus x minus big R squared dx . OK. So just a little bit of algebra there.

So this is ds . Now, OK, so now we're going-- we want to compute surface area, so we need dA , and dA is $2\pi x$ times ds . So-- OK. So that's easy to write down now that I've got ds . So this is dA equal to $2\pi r x$ over r squared minus x minus big R squared, square root of that, dx .

All right. This is dA . You know, we haven't done any calculus, actually. Oh, I guess we took a derivative somewhere. We haven't done any integration yet.

Now, to compute the surface area, we just integrate this. So we get to, you know, the calculus step of this problem. So integrate this. But of course, you know, I'm expecting a number out at the end. I'm taking a definite integral. So I need bounds. Well, what bounds do I need? Well, I'm integrating with respect to x . So I need to integrate over the relevant values of x .

What are the relevant values of x ? Well, come back to our picture over here. We have this circle. Its center is at x equals R , y equals 0 . And it has radius little r . So, you know, the relevant values of x are just from the leftmost point of the circle to the rightmost point of the circle. And so this leftmost point is just-- well, the radius is little r , so this is big R minus-- x equals big R minus little r , and the rightmost point is x equals big R plus little r .

So the bounds-- so I'm going to go up here-- so I have area is what I get when I integrate dA . And I want to integrate it from x equals big R minus little r to big R plus little r , and dA is this thing I found, just a moment ago. So this is-- well, OK. So 2π little r is a constant. I'm just going to factor that out in front. So this is 2π little r times x over the square root of little r squared minus big R minus-- sorry-- x minus big R squared. That's all under the square root. ds .

OK. So now we have to figure out how to integrate this thing. Right? So this is a little ugly. It's not horrible, though. Right? So down here we have something that's really reminiscent of one of these trig integral things we've done. Right? We've got a square root of a something squared minus a something else squared. So that, reminds, you know, what does that remind, maybe some sine substitution. Something like that. There's some there's some trig substitution waiting to happen here.

But, so OK. We could sort of shoot to do it all in one substitution. I like, my life always feels simpler when I do one little substitution at a time. And so one little substitution I could do is to simplify this x minus r thing. I could just shift this by r . So I'm just going to do a little linear substitution. I'm going to do u equals x minus big R , or I'm going to want to substitute the other way, so that's the same thing as saying x equals u plus big R . And so du equals dx . This is a simple little substitution. And I'm going to have to move the bounds, so when x is big R minus little r , then u is minus little r , and this top bound here, u is going to be equal to little r . Positive little r .

So let me just make that substitution. So area is 2π little r , integral from minus little r to plus little r . So x is u plus big R divided by, now this thing becomes square root of little r squared minus u squared du .

OK, all right. So now-- this is really kind of two separate pieces, right, for purposes of difficulty of integrating. There's the u over the square root of r squared minus u squared piece, and there's the big R over the square root of r squared minus u squared piece.

So let's think about them separately. So for this first piece, the u over the square root of little r squared minus u squared, this is something you can integrate. This is a-- you don't need a trig substitution to do that. But it's actually-- you don't need to do any work to do that. Because that function u divided by square root of little r squared minus u squared, that's an odd function. Right? This part, the denominator is even, u is odd, and we're integrating over an interval that's symmetric around the origin.

So when we integrate this u divided by this denominator part between u from minus r to r , that's just going to give 0. So I can forget about that entirely. So this is nice. And then I'll have a constant here, so I'm going to factor that out as well. So I get 2π little r big R , integral from minus r to r of du over the square root of little r squared minus u squared.

OK. Good. So far, so good. So we've got this nice simplified form for the integral. So now this is screaming out trig substitution to me. Right? There's nothing else I can do. So there's sort of two things you could do. One is you could recognize this as an integral related to arcsine, just because you remember that. The other is, you have this r squared minus u squared thing. And so r squared minus u squared, that needs a trig substitution of some sort, and the relevant trig substitution that you would want to do, is you would want to set u equal to r sine t . Why? Because then down here we'll have r squared minus r squared sine t , or if you factor out the little r , that's just 1

minus sine squared t , which is cosine squared t , and then you take a square root, and you're all good, right?

Square root of cosine squared t .

OK. So we find this trig substitution. So if we do u equals r sine t , that's good. So du , then, is r cosine t dt , and I need to change the bounds, so minus u is equal to minus r when sine of t is equal to minus 1. So that's at minus π over 2. And u is equal to r when sine of t is equal to 1. So that's a π over 2. So this is equal to 2π little r big R integral from minus π over 2 to π over 2 of r cosine t dt divided by-- OK. So then down here, we have the square root of r squared minus r squared sine squared t .

And as we said, so r squared minus r squared sine squared t , this is r squared cosine squared t . And then you take a square root, and you just get r cosine t . And so r cosine t cancels r cosine t . So the integrand here is actually 1, or 1 dt . So that's really easy to integrate. You integrate a constant, you just get-- well, the constant times the length of the integral. So this is equal to $2\pi r R$ times the constant is 1, times the length of the integral, which is another π , times π .

OK. So this is equal to 2π squared little r big R . But remember, so far we've only computed-- this is just the top half of the torus, is all we've ever talked about. So we want to get the whole torus, you just double this. So you double this, and you get the area of the whole torus is 4π squared little r big R , which is a nice formula.

So quickly, just to summarize what we've done. Standard setup. Here we did it, we're rotating around the y -axis instead of the x -axis. So this formula for dA looks a little bit different than what you saw in class, mostly. But the thing to remember is just that what goes here is the radius that this, that your little segment is traveling. And this is the circumference that it's traveling, and so this is the area of that little ribbon that it traces out.

So dA -- OK. And then we just did, you know, the sort of usual thing. For your formula, you need-- you know, you remember the formula for ds . Then you needed to find this derivative and plug it in. And so after you've done all that preparatory work then you have your integrand set up. And once you set up your integrand, you do whatever integration tools you can to hit it with.

So in our case, that was simplifying substitution, and then a nice observation that we used here to simplify this, that this part was odd. I mean, you could've-- you didn't need that observation. You could have done the problem perfectly well with another substitution there to kill off that first part. And then, OK, and then a trig substitution, remembering that this is an arcsine, or related to an arcsine, to finish it off.

And is there anything else I want to say? I think that's-- oh, yes. There's one other thing I wanted to say, which is that we could have done this slightly differently. Which is, here we solved for y explicitly in terms of x . And it would have been possible to carry this through using implicit differentiation instead. It would have actually simplified-- I

would have had to write this square root of little r squared minus x minus big R quantity squared fewer times if I'd done implicit differentiation, starting from this implicit equation relating x and y . You have to be a little careful there, whether you're doing the whole circle all at once, or just the top half or the bottom half. But you could do it with implicit differentiation instead, and maybe save yourself a little bit of messy-looking arithmetic.

So I think I'll end there.