

The following content is provided under a Creative Commons license. Your support will help MIT OpenCourseWare continue to offer high quality educational resources for free. To make a donation or to view additional materials from hundreds of MIT courses, visit MIT OpenCourseWare at [ocw.mit.edu](http://ocw.mit.edu).

**PROFESSOR:** Today we're going to continue our discussion of parametric curves. I have to tell you about arc length. And let me remind me where we left off last time. This is parametric curves, continued. Last time, we talked about the parametric representation for the circle. Or one of the parametric representations for the circle. Which was this one here. And first we noted that this does parameterize, as we say, the circle. That satisfies the equation for the circle. And it's traced counterclockwise. The picture looks like this. Here's the circle. And it starts out here at  $t = 0$  and it gets up to here at time  $t = \pi / 2$ . So now I have to talk to you about arc length. In this parametric form. And the results should be the same as arc length around this circle ordinarily. And we start out with this basic differential relationship.  $ds^2$  is  $dx^2 + dy^2$ . And then I'm going to take the square root, divide by  $dt$ , so the rate of change with respect to  $t$  of  $s$  is going to be the square root. Well, maybe I'll write it without dividing. Just write it as  $ds$ . So this would be  $(dx/dt)^2 + (dy/dt)^2, dt$ .

So this is what you get formally from this equation. If you take its square roots and you divide by  $dt$  squared in the-- inside the square root, and you multiply by  $dt$  outside, so that those cancel. And this is the formal connection between the two. We'll be saying just a few more words in a few minutes about how to make sense of that rigorously. Alright so that's the set of formulas for the infinitesimal, the differential of arc length. And so to figure it out, I have to differentiate  $x$  with respect to  $t$ . And remember  $x$  is up here. It's defined by a  $\cos t$ , so its derivative is  $-a \sin t$ . And similarly,  $dy/dt = a \cos t$ .

And so I can plug this in. And I get the arc length element, which is the square root of  $(-a \sin t)^2 + (a \cos t)^2, dt$ . Which just becomes the square root of  $a^2, dt$ , or  $a dt$ . Now, I was about to divide by  $t$ . Let me do that now. We can also write the rate of change of arc length with respect to  $t$ . And that's  $a$ , in this case. And this gets interpreted as the speed of the particle going around. So not only, let me trade these two guys, not only do we have the direction is counterclockwise, but we also have that the speed is, if you like, it's uniform. It's constant speed. And the rate is  $a$ . So that's  $ds/dt$ . Travelling around.

And that means that we can play around with the speed. And I just want to point out-- So the standard thing, what you'll have to get used to, and this is a standard presentation, you'll see this everywhere. In your physics classes and your other math classes, if you want to change the speed, so a new speed going around this would be, if I set up the equations this way. Now I'm tracing around the same circle. But the speed is going to turn out to be, if you figure it out, there'll be an extra factor of  $k$ . So it'll be  $ak$ . That's what we'll work out to be the speed. Provided  $k$  is positive and  $a$  is positive. So we're making these conventions. The constants that we're using are positive.

Now, that's the first and most basic example. The one that comes up constantly. Now, let me just make those comments about notation that I wanted to make. And we've been treating these squared differentials here for a little while and I just want to pay attention a little bit more carefully to these manipulations. And what's allowed and what's not. And what's justified and what's not. So the basis for this was this approximate calculation that we had, that  $(\Delta s)^2$  was  $(\Delta x)^2 + (\Delta y)^2$ . This is how we justified the arc length formula before. And let me just show you that the formula that I have up here, this basic formula for arc length in the parametric form, follows just as the other one did. And now I'm going to do it slightly more rigorously.

I do the division really in disguise before I take the limit of the infinitesimal. So all I'm really doing is I'm doing this. Dividing through by this, and sorry this is still approximately equal. So I'm not dividing by something that's 0 or infinitesimal. I'm dividing by something nonzero. And here I have  $((\Delta x)/(\Delta t))^2 + ((\Delta y)/(\Delta t))^2$ . And then in the limit, I have  $ds/dt$  is equal to the square root of this guy. Or, if you like, the square of it, so. So it's legal to divide by something that's almost 0 and then take the limit as we go to 0. This is really what derivatives are all about. That we get a limit here. As the denominator goes to 0. Because the numerator's going to 0 too. So that's the notation.

And now I want to warn you, maybe just a little bit, about misuses, if you like, of the notation. We don't do absolutely everything this way. This expression that came up with the squares, you should never write it as this. This, put it on the board but very quickly, never. OK. Don't do that. We use these square differentials, but we don't do it with these ratios here. But there was another place which is slightly confusing. It looks very similar, where we did use the square of the differential in a denominator. And I just want to point out to you that it's different. It's not the same. And it is OK. And that was this one. This thing here. This is a second derivative, it's

something else. And it's got a  $dt^2$  in the denominator. So it looks rather similar. But what this represents is the quantity  $d/dt$  squared. And you can see the squares came in. And squared the two expressions. And then there's also an  $x$  over here.

So that's legal. Those are notations that we do use. And we can even calculate this. It has a perfectly good meaning. It's the same as the derivative with respect to  $t$  of the derivative of  $x$ , which we already know was minus sine-- sorry, a  $\sin t$ , I guess. Not this example, but the previous one. Up here. So the derivative is this and so I can differentiate a second time. And I get  $-a \cos t$ . So that's a perfectly legal operation. Everything in there makes sense. Just don't use that. There's another really unfortunate thing, right which is that the 2 creeps in funny places with sines. You have sine squared. It would be out here, it comes up here for some strange reason. This is just because typographers are lazy or somebody somewhere in the history of mathematical typography decided to let the 2 migrate. It would be like putting the 2 over here. There's inconsistency in mathematics, right. We're not perfect and people just develop these notations. So we have to live with them. The ones that people accept as conventions.

The next example that I want to give you is just slightly different. It'll be a non-constant speed parameterization. Here  $x = 2 \sin t$ . And  $y$  is, say,  $\cos t$ . And let's keep track of what this one does. Now, this is a skill which I'm going to ask you about quite a bit. And it's one of several skills. You'll have to connect this with some kind of rectangular equation. An equation for  $x$  and  $y$ . And we'll be doing a certain amount of this today. In another context. Right here, to see the pattern, we know that the relationship we're going to want to use is that  $\sin^2 + \cos^2 = 1$ . So in fact the right thing to do here is to take  $1/4 x^2 + y^2$ . And that's going to turn out to be  $\sin^2 t + \cos^2 t$ . Which is 1. So there's the equation. Here's the rectangular equation for this parametric curve. And this describes an ellipse.

That's not the only information that we can get here. The other information that we can get is this qualitative information of where we start, where we're going, the direction. It starts out, I claim, at  $t = 0$ . That's when  $t = 0$ , this is  $(2 \sin 0, \cos 0)$ , right?  $(2 \sin 0, \cos 0)$  is equal to the point  $(0, 1)$ . So it starts up up here. At  $(0, 1)$ . And then the next little place, so this is one thing that certainly you want to do.  $t = \pi/2$  is maybe the next easy point to plot. And that's going to be  $(2 \sin(\pi/2), \cos(\pi/2))$ . And that's just  $(2, 0)$ . And so that's over here somewhere. This is  $(2, 0)$ . And we know it travels along the ellipse. And we know the minor axis is 1, and the major axis is 2, so it's doing this.

So this is what happens at  $t = 0$ . This is where we are at  $t = \pi/2$ . And it continues all the way around, etc. To the rest of the ellipse. This is the direction. So this one happens to be clockwise.

Alright, now let's keep track of its speed. Let's keep track of the speed, and also the arc length. So the speed is the square root of the derivatives here. That would be  $(2 \cos t)^2 + (\sin t)^2$ . And the arc length is what? Well, if we want to go all the way around, we need to know that that takes a total of  $2\pi$ . So  $0$  to  $2\pi$ . And then we have to integrate  $ds$ , which is this expression, or  $ds/dt, dt$ . So that's the square root of  $4 \cos^2 t + \sin^2 t, dt$ .

The bad news, if you like, is that this is not an elementary integral. In other words, no matter how long you try to figure out how to antidifferentiate this expression, no matter how many substitutions you try, you will fail. That's the bad news. The good news is this is not an elementary integral. It's not an elementary integral. Which means that this is the answer to a question. Not something that you have to work on. So if somebody asks you for this arc length, you stop here. That's the answer, so it's actually better than it looks. And we'll try to-- I mean, I don't expect you to know already what all of the integrals are that are impossible. And which ones are hard and which ones are easy. So we'll try to coach you through when you face these things. It's not so easy to decide. I'll give you a few clues, but. OK. So this is the arc length.

Now, I want to move on to the last thing that we did. Last type of thing that we did last time. Which is the surface area. And yeah, question.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** The question, this is a good question. The question is, when you draw the ellipse, do you not take into account what  $t$  is. The answer is that this is in disguise. What's going on here is we have a trouble with plotting in the plane what's really happening. So in other words, it's kind of in trouble. So the point is that we have two functions of  $t$ , not one.  $x(t)$  and  $y(t)$ . So one thing that I can do if I plot things in the plane. In other words, the main point to make here is that we're not talking about the situation  $y$  is a function of  $x$ . We're out of that realm now. We're somewhere in a different part of the universe in our thought. And you should drop this point of view. So this depiction is not  $y$  as a function of  $x$ . Well, that's obvious because there are two values here, as opposed to one. So we're in trouble with that. And we have that background parameter, and that's exactly why we're using it. This parameter  $t$ . So that we can depict the

entire curve. And deal with it as one thing.

So since I can't really draw it, and since  $t$  is nowhere on the map, you should sort of imagine it as time, and there's some kind of trajectory which is travelling around. And then I just labelled a couple of the places. If somebody asked you to draw a picture of this, well, I'll tell you exactly where you need the picture in just one second, alright. It's going to come up right now in surface area. But otherwise, if nobody asks you to, you don't even have to put down  $t = 0$  and  $t = \pi / 2$  here. Because nobody demanded it of you. Another question.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** So, another very good question which is exactly connected to this picture. So how is it that we're going to use the picture, and how is it we're going to use the notion of the  $t$ . The question was, why is this from  $t = 0$  to  $t = 2\pi$ ? That does use the  $t$  information on this diagram. The point is, we do know that  $t$  starts here. This is  $\pi / 2$ , this is  $\pi$ , this is  $3\pi / 2$ , and this is  $2\pi$ . When you go all the way around once, it's going to come back to itself. These are periodic functions of period  $2\pi$ . And they come back to themselves exactly at  $2\pi$ . And so that's why we know in order to get around once, we need to go from  $0$  to  $2\pi$ . And the same thing is going to come up with surface area right now. That's going to be the issue, is what range of  $t$  we're going to need when we compute the surface area.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** In a question, what you might be asked is what's the rectangular equation for a parametric curve? So that would be  $\frac{1}{4}x^2 + y^2 = 1$ . And then you might be asked, plot it. Well, that would be a picture of the ellipse. OK, those are types of questions that are legal questions.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** The question is, do I need to know any specific formulas? Any formulas that you know and remember will help you. They may be of limited use. I'm not going to ask you to memorize anything except, I guarantee you that the circle is going to come up. Not the ellipse, the circle will come up everywhere in your life. So at least at MIT, your life at MIT. We're very round here. Yeah, another question.

**STUDENT:** I'm just a tiny bit confused back to the basics. This is more a question from yesterday, I guess. But when you have your original  $ds^2 = dx^2 + dy^2$ , and then you integrate that to get arc length, how are you, the integral has  $dx$ 's and  $dy$ 's. So how are you just integrating with

respect to  $dx$ ?

**PROFESSOR:**

OK, the question is how are we just integrating with respect to  $x$ ? So this is a question which goes back to last time. And what is it with arc length. So. I'm going to have to answer that question in connection with what we did today. So this is a subtle question. But I want you to realize that this is actually an important conceptual step here. So shhh, everybody, listen.

If you're representing one-dimensional objects, which are curves, maybe, in space. Or in two dimensions. When you're keeping track of arc length, you're going to have to have an integral which is with respect to some variable. But that variable, you get to pick. And we're launching now into this variety of choices of variables with respect to which you can represent something. Now, there are some disadvantages on the circle to representing things with respect to the variable  $x$ . Because there are two points on the circle here. On the other hand, you actually can succeed with half the circle. So you can figure out the arc length that way. And then you can set it up as an integral  $dx$ . But you can also set it up as an integral with respect to any parameter you want. And the uniform parameter is perhaps the easiest one. This one is perhaps the easiest one.

And so now the thing that's strange about this perspective - and I'm going to make this point later in the lecture as well - is that the letters  $x$  and  $y$ -- As I say, you should drop this notion that  $y$  is a function of  $x$ . This is what we're throwing away at this point. What we're thinking of is, you can describe things in terms of any coordinate you want. You just have to say what each one is in terms of the others. And these  $x$  and  $y$  over here are where we are in the Cartesian coordinate system. They're not-- And in this case they're functions of some other variable. Some other variable. So they're each functions. So the letters  $x$  and  $y$  just changed on you. They mean something different.  $x$  is no longer the variable. It's the function. Right?

You're going to have to get used to that. That's because we run out of letters. And we kind of want to use all of them the way we want. I'll say some more about that later.

So now I want to do this surface area example. I'm going to just take the surface area of the ellipsoid. The surface of the ellipsoid formed by revolving this previous example, which was Example 2. Around the  $y$ -axis. So we want to set up that surface area integral here for you. Now, I remind you that the area element looks like this. If you're revolving around the  $y$ -axis, that means you're going around this way and you have some curve. In this case it's this piece of an ellipse. If you sweep it around you're going to get what's called an ellipsoid. And there's a

little chunk here, that you're wrapping around. And the important thing you need besides this  $ds$ , this arc length piece over here, is the distance to the axis. So that's this horizontal distance here. I'll draw it in another color. And that horizontal distance now has a name. And this is, again, the virtue of this coordinate system. The  $t$  is something else. This has a name. This distance has a name. This distance is called  $x$ .

And it even has a formula. Its formula is  $2 \sin t$ . In terms of  $t$ . So the full formula up for the integral here is, I have to take the circumference when I spin this thing around. And this little arc length element. So I have here  $2 \pi$  times  $2 \sin t$ . That's the  $x$  variable here. And then I have here  $ds$ , which is kind of a mess. So unfortunately I don't quite have room for it. Plan ahead. Square root of  $4 \cos^2 t + \sin^2 t$ , is that what it was,  $dt$ . Alright, I guess I squeezed it in there. So that was the arc length, which I re-copied from this board above. That was the  $ds$  piece. It's this whole thing including the  $dt$ . That's the answer except for one thing. What else do we need? We don't just need the integrand, this is half of setting up an integral. The other half of setting up an integral is the limits. We need specific limits here. Otherwise we don't have a number that we can get out.

So we now have to think about what the limits are. And maybe somebody can see. It has something to do with this diagram of the ellipse over here. Can somebody guess what it is?  $0$  to  $\pi$ . Well, that was quick. That's it. Because we go from the top to the bottom, but we don't want to continue around. We don't want to go from  $0$  to  $2\pi$ , because that would be duplicating what we're going to get when we spin around. And we know that we start at  $0$ . It's interesting because it descends when you change variables to think of it in terms of the  $y$  variable it's going the opposite way. But anyway, just one piece of this is what we want.

So that's this setup. And now I claim that this is actually a doable integral. However, it's long. I'm going to spare you, I'll just tell you how you would get started. You would use the substitution  $u = \cos t$ . And then the  $du$  is going to be  $-\sin t dt$ . But then, unfortunately, there's a lot more. There's another trig substitution with some other multiple of the cosine and so forth. So it goes on and on. If you want to check it yourself, you can. There's an inverse trig substitution which isn't compatible with this one. But it can be done. Calculated. In elementary terms. Yeah, another question.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** So, if you get this on an exam, I'm going to have to coach you through it. Either I'm going to

have to tell you don't evaluate it or, you're going to have to work really hard. Or here's the first step, and then the next step is, keep on going. Or something. I'll have to give you some cues. Because it's quite long. This is way too long for an exam, this particular one. OK. It's not too long for a problem set. This is where I would leave you off if I were giving it to you on a problem set. Just to give you an idea of the order of magnitude. Whereas one of the ones that I did yesterday, I wouldn't even give you on a problem set, it was so long.

So now, our next job is to move on to polar coordinates. Now, polar coordinates involve the geometry of circles. As I said, we really love circles here. We're very round. Just as I love 0, the rest of the Institute loves circles. So we're going to do that right now.

What we're going to talk about now is polar coordinates. Which are set up in the following way. It's a way of describing the points in the plane. Here is a point in a plane, and here's what we think of as the usual x-y axes. And now this point is going to be described by a different pair of coordinates, different pair of numbers. Namely, the distance to the origin. And the second parameter here, second number here, is this angle theta. Which is the angle of ray from origin with the horizontal axis. So that's what it is in language. And you should put this in quotation marks, because it's not a perfect match. This is geometrically what you should always think of, but the technical details involve dealing directly with formulas.

The first formula is the formula for x. And this is the fundamental, these two are the fundamental ones. Namely,  $x = r \cos \theta$ . The second formula is the formula for y, which is  $r \sin \theta$ . So these are the unambiguous definitions of polar coordinates. This is it. And this is the thing from which all other almost correct statements almost follow. But this is the one you should trust always. This is the unambiguous statement.

So let me give you an example something that's close to being a good formula and is certainly useful in its way. Namely, you can think of r as being the square root of  $x^2 + y^2$ . That's easy enough to derive, it's the distance to the origin. That's pretty obvious. And the formula for theta, which you can also derive, which is that it's the inverse tangent of  $y/x$ . However, let me just warn you that these formulas are slightly ambiguous. So somewhat ambiguous. In other words, you can't just apply them blindly. You actually have to look at a picture in order to get them right. In particular, r could be plus or minus here. And when you take the inverse tangent, there's an ambiguity between, it's the same as the inverse tangent of  $(-y)/(-x)$ . So these minus signs are a plague on your existence. And you're not going to get a completely unambiguous answer out of these formulas without paying attention to the diagram. On the



other hand, the formula up in the box there always works. So when people mean polar coordinates, they always mean that. And then they have conventions, which sometimes match things up with the formulas over on this next board.

Let me give you various examples here first. But maybe first I should I should draw the two coordinate systems. So the coordinate system that we're used to is the rectangular coordinate system. And maybe I'll draw it in orange and green here. So these are the coordinate lines  $y = 0$ ,  $y = 1$ ,  $y = 2$ . That's how the coordinate system works. And over here we have the rest of the coordinate system. And this is the way we're thinking of  $x$  and  $y$  now. We're no longer thinking of  $y$  as a function of  $x$  and  $x$  as a function of  $y$ , we're thinking of  $x$  as a label of a place in a plane. And  $y$  as a label of a place in a plane.

So here we have  $x = 0$ ,  $x = 1$ ,  $x = 2$ , etc. Here's  $x = -1$ . So forth. So that's what the rectangular coordinate system looks like. And now I should draw the other coordinate system that we have. Which is this guy here. Well, close enough. And these guys here. Kind of this bulls-eye or target operation. And this one is, say,  $\theta = \pi/2$ . This is  $\theta = 0$ . This is  $\theta = -\pi/4$ . For instance, so I've just labeled for you three of the rays on this diagram. It's kind of like a radar screen. And then in pink, this is maybe  $r = 2$ , the radius 2. And inside is  $r = 1$ . So it's a different coordinate system for the plane. And again, the letter  $r$  represents measuring how far we are from the origin. The  $\theta$  represents something about the angle, which ray we're on. And they're just two different variables. And this is a very different kind of coordinate system.

OK so, our main job is just to get used to this. For now. You will be using this a lot in 18.02. It's very useful in physics. And our job is just to get started with it. And so, let's try a few examples here. Tons of examples. We'll start out very slow. If you have  $(x, y) = (1, -1)$ , that's a point in the plane. I can draw that point. It's down here, right? This is  $-1$  and this is  $1$ , and here's my point,  $(1, -1)$ . I can figure out what the representative is of this in polar coordinates. So in polar coordinates, there are actually a bunch of choices here.

First of all, I'll tell you one choice. If I start with the angle horizontally, I wrap all the way around, that would be to this ray here-- Let's do it in green again. Alright, I labeled it actually as  $-\pi/4$ , but another way of looking at it is that it's this angle here. So that would be  $r = \sqrt{2}$ .  $\theta = 7\pi/4$ . So that's one possibility of the angle and the distance. I know the distance is a square root of 2, that's not hard.

Another way of looking at it is the way which was suggested when I labeled this with a negative

angle. And that would be  $r = \text{square root of } 2$ ,  $\theta = -\pi/4$ . And these are both legal. These are perfectly legal representatives. And that's what I meant by saying that these representations over here are somewhat ambiguous. There's more than one answer to this question, of what the polar representation is.

A third possibility, which is even more dicey but also legal, is  $r = \text{minus square root of } 2$ .  $\theta = 3\pi/4$ . Now, what that corresponds to doing is going around to here. We're pointing out  $3/4 \pi$  direction. But then going negative square root of 2 distance. We're going backwards. So we're landing in the same place. So this is also legal. Yeah.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** The question is, don't the radiuses have to be positive because they represent a distance to the origin? The answer is I lied to you here. All of these things that I said are wrong, except for this. Which is the rule for what polar coordinates mean. So it's maybe plus or minus the distance, is what it is always. I try not to lie to you too much, but I do succeed. Now, let's do a little bit more practice here.

There are some easy examples, which I will run through very quickly.  $r = a$ , we already know this is a circle. And the  $\theta = \text{constant}$  is a ray. However, this involves an implicit assumption, which I want to point out to you. So this is Example 3.  $\theta = \text{constant}$  is a ray. But this implicitly assumes  $0 \leq r < \infty$ . If you really wanted to allow  $-\infty < r < \infty$  in this example, you would get a line. Gives the whole line. It gives everything behind. So you go out on some ray, you go backwards on that ray and you get the whole line through the origin, both ways. If you allow  $r$  going to minus infinity as well.

So the typical conventions, so here are the typical conventions. And you will see people assume this without even telling you. So you need to watch out for it. The typical conventions are certainly this one, which is a nice thing to do. Pretty much all the time, although not all the time. Most of the time. And then you might have  $\theta$  ranging from  $-\pi$  to  $\pi$ , so in other words symmetric around 0. Or, another very popular choice is this one.  $\theta \geq 0$  and strictly less than  $2\pi$ . So these are the two typical ranges in which all of these variables are chosen. But not always. You'll find that it's not consistent.

As I said, our job is to get used to this. And I need to work up to some slightly more complicated examples. Some of which I'll give you on next Tuesday. But let's do a few more. So, I guess this is Example 4. Example 4, I'm going to take  $y = 1$ . That's awfully simple in

rectangular coordinates. But interestingly, you might conceivably want to deal with it in polar coordinates. If you do, so here's how you make the translation. But this translation is not so terrible. What you do is, you plug in  $y = r \sin(\theta)$ . That's all you have to do. And so that's going to be equal to 1. And that's going to give us our polar equation. The polar equation is  $r = 1 / \sin(\theta)$ . There it is. And let's draw a picture of it. So here's a picture of the line  $y = 1$ . And now we see that if we take our rays going out from here, they collide with the line at various lengths. So if you take an angle,  $\theta$ , here there'll be a distance  $r$  corresponding to that and you'll hit this in exactly one spot. For each  $\theta$  you'll have a different radius. And it's a variable radius. It's given by this formula here. And so to trace this line out, you actually have to realize that there's one more thing involved. Which is the possible range of  $\theta$ . Again, when you're doing integrations you're going to need to know those limits of integration. So you're going to need to know this. The range here goes from  $\theta = 0$ , that's sort of when it's out at infinity. That's when the denominator is 0 here. And it goes all the way to  $\pi$ . Swing around just one half-turn. So the range here is  $0 < \theta < \pi$ . Yeah, question.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:** The question is, is it typical to express  $r$  as a function of  $\theta$ , or vice versa, or does it matter? The answer is that for the purposes of this course, we're almost always going to be writing things in this form.  $r$  as a function of  $\theta$ . And you can do whatever you want. This turns out to be what we'll be doing in this course, exclusively. As you'll see when we get to other examples, it's the traditional sort of thing to do when you're thinking about observing a planet or something like that. You see the angle, and then you guess far away it is. But it's not necessary. The formulas are often easier this way. For the examples that we have. Because it's usually a trig function of  $\theta$ . Whereas the other way, it would be an inverse trig function. So it's an uglier expression. As you can see. The real reason is that we choose this thing that's easier to deal with.

So now let me give you a slightly more complicated example of the same type. Where we use a shortcut. This is a standard example. And it comes up a lot. And so this is an off-center circle. A circle is really easy to describe, but not necessarily if the center is on the rim of the circle. So that's a different problem. And let's do this with a circle of radius  $a$ . So this is the point  $(a, 0)$  and this is  $(2a, 0)$ . And actually, if you know these two numbers, you'll be able to remember the result of this calculation. Which you'll do about five or six times and then finally you'll memorize it during 18.02 when you will need it a lot. So this is a standard calculation

here. So the starting place is the rectangular equation. And we're going to pass to the polar representation. The rectangular representation is  $(x-a)^2 + y^2 = a^2$ . So this is a circle centered at  $(a, 0)$  of radius  $a$ .

And now, if you like, the slow way of doing this would be to plug in  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ . The way I did in this first step. And that works perfectly well. But I'm going to do it more quickly than that. Because I can sort of see in advance how it's going to work. I'm just going to expand this out. And now I see the  $a^2$ 's cancel. And not only that, but  $x^2 + y^2 = r^2$ . So this becomes  $r^2$ . That's  $x^2 + y^2 - 2ax = 0$ .

The  $r$  came from the fact that  $r^2 = x^2 + y^2$ . So I'm doing this the rapid way. You can do it by plugging in, as I said.  $r$  equals-- So now that I've simplified it, I am going to use that procedure. I'm going to plug in. So here I have  $r^2 - 2ar \cos(\theta) = 0$ . I just plugged in for  $x$ . As I said, I could have done that at the beginning. I just simplified first. And now, this is the same thing as  $r^2 = 2ar \cos(\theta)$ . And we're almost done. There's a boring part of this equation, which is  $r = 0$ . And then there's, if I divide by  $r$ , there's the interesting part of the equation. Which is this. So this is or  $r = 0$ . Which is already included in that equation anyway.

So I'm allowed to divide by  $r$  because in the case of  $r = 0$ , this is represented anyway.

Question.

**STUDENT:** [INAUDIBLE]

**PROFESSOR:**  $r = 0$  is just one case. That is, it's the union of these two. It's both. Both are possible. So  $r = 0$  is one point on it. And this is all of it. So we can just ignore this. So now I want to say one more important thing. You need to understand the range of this. So wait a second and we're going to figure out the range here. The range is very important, because otherwise you'll never be able to integrate using this representation here. So this is the representation. But notice when  $\theta = 0$ , we're out here at  $2a$ . That's consistent, and that's actually how you remember this factor  $2a$  here. Because if you remember this picture and where you land when  $\theta = 0$ . So that's the  $\theta = 0$  part. But now as I tip up like this, you see that when we get to vertical, we're done. With the circle. It's gotten shorter and shorter and shorter, and at  $\theta = \pi/2$ , we're down at  $0$ . Because that's  $\cos(\pi/2) = 0$ . So it swings up like this. And it gets up to  $\pi/2$ . Similarly, we swing down like this. And then we're done. So the range is  $-\pi/2 < \theta < \pi/2$ . Or, if you want to throw in the  $r = 0$  case, you can throw in this, this is repeating, if you like, at the ends. So this is the range of this circle. And let's see. Next time we'll figure out area in polar

coordinates.