18.02 Practice Exam 2 A – Solutions

Problem 1.

a) $\nabla f = (y - 4x^3)\,\hat{\imath} + x\,\hat{\jmath}$; at P, $\nabla f = \langle -3, 1\rangle$.

b) $\Delta w \simeq -3\,\Delta x + \Delta y$.

Problem 2.

a) By measuring, $\Delta h = 100$ for $\Delta s \simeq 500$, so $\left(\dfrac{dh}{ds}\right)_{\hat{u}} \simeq \dfrac{\Delta h}{\Delta s} \simeq .2$.

b) Q is the northernmost point on the curve $h = 2200$; the vertical distance between consecutive level curves is about 1/3 of the given length unit, so $\dfrac{\partial h}{\partial y} \simeq \dfrac{\Delta h}{\Delta y} \simeq \dfrac{-100}{1000/3} \simeq -.3$.

Problem 3.

$f(x,y,z) = x^3y + z^2 = 3$: the normal vector is $\nabla f = \langle 3x^2y, x^3, 2z\rangle = \langle 3, -1, 4\rangle$. The tangent plane is $3x - y + 4z = 4$.

Problem 4.

a) The volume is $xyz = xy(1-x^2-y^2) = xy - x^3y - xy^3$. Critical points: $f_x = y - 3x^2y - y^3 = 0$, $f_y = x - x^3 - 3xy^2 = 0$.

b) Assuming $x > 0$ and $y > 0$, the equations can be rewritten as $1 - 3x^2 - y^2 = 0$, $1 - x^2 - 3y^2 = 0$. Solution: $x^2 = y^2 = 1/4$, i.e. $(x,y) = (1/2, 1/2)$.

c) $f_{xx} = -6xy = -3/2$, $f_{yy} = -6xy = -3/2$, $f_{xy} = 1 - 3x^2 - 3y^2 = -1/2$. So $f_{xx}f_{yy} - f_{xy}^2 > 0$, and $f_{xx} < 0$, it is a local maximum.

d) The maximum of f lies either at $(1/2, 1/2)$, or on the boundary of the domain or at infinity. Since $f(x,y) = xy(1-x^2-y^2)$, $f = 0$ when either $x \to 0$ or $y \to 0$, and $f \to -\infty$ when $x \to \infty$ or $y \to \infty$ (since $x^2 + y^2 \to \infty$). So the maximum is at $(x,y) = (\frac{1}{2}, \frac{1}{2})$, where $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{8}$.

Problem 5.

a) $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z = 1$: one must solve the Lagrange multiplier equation $\nabla f = \lambda \nabla g$, i.e. $yz = 2\lambda x$, $xz = 2\lambda y$, $xy = \lambda$, and the constraint equation $x^2 + y^2 + z = 1$.

b) Dividing the first two equations $yz = 2\lambda x$ and $xz = 2\lambda y$ by each other, we get $y/x = x/y$, so $x^2 = y^2$; since $x > 0$ and $y > 0$ we get $y = x$. Substituting this into the Lagrange multiplier equations, we get $z = 2\lambda$ and $x^2 = \lambda$. Hence $z = 2x^2$, and the constraint equation becomes $4x^2 = 1$, so $x = \frac{1}{2}$, $y = \frac{1}{2}$, $z = \frac{1}{2}$.

Problem 6.

$\dfrac{\partial w}{\partial x} = f_u u_x + f_v v_x = y\, f_u + \dfrac{1}{y}\, f_v$. $\quad \dfrac{\partial w}{\partial y} = f_u u_y + f_v v_y = x\, f_u - \dfrac{x}{y^2}\, f_v$.

Problem 7.

Using the chain rule: $\left(\dfrac{\partial w}{\partial z}\right)_y = \dfrac{\partial w}{\partial x}\left(\dfrac{\partial x}{\partial z}\right)_y = 3x^2y\left(\dfrac{\partial x}{\partial z}\right)_y$. To find $\left(\dfrac{\partial x}{\partial z}\right)_y$, differentiate the relation $x^2y + xz^2 = 5$ w.r.t. z holding y constant: $(2xy + z^2)\left(\dfrac{\partial x}{\partial z}\right)_y + 2xz = 0$, so $\left(\dfrac{\partial x}{\partial z}\right)_y = \dfrac{-2xz}{2xy + z^2}$. Therefore $\left(\dfrac{\partial w}{\partial z}\right)_y = \dfrac{-6x^3yz}{2xy + z^2}$. At $(x,y,z) = (1,1,2)$ this is equal to -2.