18.02 Multivariable Calculus
Fall 2007

18.02 Practice Exam 4B – Solutions
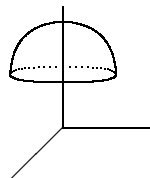
Problem 1.

$$\int_0^{\pi/2} \int_0^1 \int_0^1 r^2 \, r \, dz \, dr \, d\theta.$$
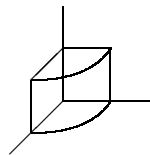
Problem 2.

a) sphere: $\rho = 2a \cos \phi$.   b) plane: $\rho = a \sec \phi$.

c) $\displaystyle\int_0^{2\pi} \int_0^{\pi/4} \int_{a \sec \phi}^{2a \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$

Problem 3.

a) $\frac{\partial}{\partial y}(2xy + z^3) = 2x = \frac{\partial}{\partial x}(x^2 + 2yz);$ $\frac{\partial}{\partial z}(2xy + z^3) = 3z^2 = \frac{\partial}{\partial x}(y^2 + 3xz^2 - 1);$

$\frac{\partial}{\partial z}(x^2 + 2yz) = 2y = \frac{\partial}{\partial y}(y^2 + 3xz^2 - 1);$ so $\vec{F}$ is conservative.
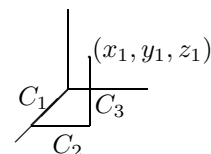
b) <u>Method 1:</u> $f(x, y, z) = \int_{C_1 + C_2 + C_3} \vec{F} \cdot d\vec{r};$

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{x_1} (2xy + z^3) \, dx = \int_0^{x_1} 0 \, dx = 0$ $(y = 0, \ z = 0)$

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{y_1} (x^2 + 2yz) \, dy = \int_0^{y_1} x_1^2 \, dy = x_1^2 y_1$ $(x = x_1, \ z = 0)$

$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^{z_1} (y^2 + 3xz^2 - 1) \, dz = \int_0^{z_1} (y_1^2 + 3x_1 z^2 - 1) \, dz = y_1^2 z_1 + x_1 z_1^3 - z_1$ $(x = x_1, \ y = y_1)$

So $f(x, y, z) = x^2 y + y^2 z + xz^3 - z + c.$

<u>Method 2:</u> $\frac{\partial f}{\partial x} = 2xy + z^3$, so $f(x, y, z) = x^2 y + xz^3 + g(y, z).$

$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 + 2yz$, so $\frac{\partial g}{\partial y} = 2yz.$

   Therefore $g(y, z) = y^2 z + h(z)$, and $f(x, y, z) = x^2 y + xz^3 + y^2 z + h(z).$

$\frac{\partial f}{\partial z} = 3xz^2 + y^2 + h'(z) = y^2 + 3xz^2 - 1$, so $h'(z) = -1.$

   Therefore $h(z) = -z + c$, and $f(x, y, z) = x^2 y + xz^3 + y^2 z - z + c.$

Problem 4.

a) S is the graph of $z = f(x, y) = 1 - x^2 - y^2$, so $\hat{n} \, dS = \langle -f_x, -f_y, 1 \rangle \, dA = \langle 2x, 2y, 1 \rangle \, dA.$

Therefore $\iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \langle x, y, 2(1 - z) \rangle \cdot \langle 2x, 2y, 1 \rangle \, dA = \iint_S 2x^2 + 2y^2 + 2(1 - z) \, dA = \iint_S 4x^2 + 4y^2 \, dA$ (since $z = 1 - x^2 - y^2$).

Shadow = unit disc $x^2 + y^2 \le 1$; switching to polar coordinates, we have

$\iint_S \vec{F} \cdot \hat{n} \, dS = \int_0^{2\pi} \int_0^1 4r^2 \, r \, dr \, d\theta = \int_0^{2\pi} \left[r^4 \right]_0^1 d\theta = 2\pi.$

b) Let T = unit disc in the xy-plane, with normal vector pointing down ($\hat{n} = -\hat{k}$). Then

$\iint_T \vec{F} \cdot \hat{n} \, dS = \iint_T \langle x, y, 2 \rangle \cdot (-\hat{k}) \, dS = \iint_T -2 \, dS = -2 \, \text{Area} = -2\pi.$ By divergence theorem,

$\iint_{S+T} \vec{F} \cdot \hat{n} \, dS = \iiint_D \text{div} \, \vec{F} \, dV = 0$, since $\text{div} \, \vec{F} = 1 + 1 - 2 = 0$. Therefore $\iint_S = -\iint_T = +2\pi.$

Problem 5.

a) $\text{curl} \, \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -2xz & 0 & y^2 \end{vmatrix} = 2y\hat{i} - 2x\hat{j}.$

b) On the unit sphere, $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$, so $\text{curl} \, \vec{F} \cdot \hat{n} = \langle 2y, -2x, 0 \rangle \cdot \langle x, y, z \rangle = 2xy - 2xy = 0$; therefore $\iint_R \text{curl} \, \vec{F} \cdot \hat{n} \, dS = 0.$

c) By Stokes, $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl} \, \vec{F} \cdot \hat{n} \, dS$, where R is the region delimited by C on the unit sphere. Using the result of b), we get $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl} \, \vec{F} \cdot \hat{n} \, dS = 0.$