

## V11. Line Integrals in Space

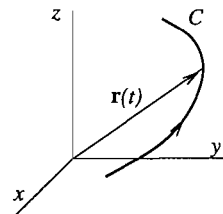
### 1. Curves in space.

In order to generalize to three-space our earlier work with line integrals in the plane, we begin by recalling the relevant facts about parametrized space curves.

In 3-space, a vector function of one variable is given as

$$(1) \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} .$$

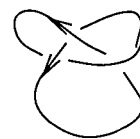
It is called *continuous* or *differentiable* or *continuously differentiable* if respectively  $x(t)$ ,  $y(t)$ , and  $z(t)$  all have the corresponding property. By placing the vector so that its tail is at the origin, its head moves along a curve  $C$  as  $t$  varies. This curve can be described therefore either by its position vector function (1), or by the three parametric equations



$$(2) \quad x = x(t), \quad y = y(t), \quad z = z(t) .$$

The curves we will deal with will be finite, connected, and piecewise smooth; this means that they have finite length, they consist of one piece, and they can be subdivided into a finite number of smaller pieces, each of which is given as the position vector of a continuously differentiable function (i.e., one whose derivative is continuous).

In addition, the curves will be *oriented*, or *directed*, meaning that an arrow has been placed on them to indicate which direction is considered to be the positive one. The curve is called *closed* if a point  $P$  moving on it always in the positive direction ultimately returns to its starting position, as in the accompanying picture.



The derivative of  $\mathbf{r}(t)$  is defined in terms of components by

$$(3) \quad \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} .$$

If the parameter  $t$  represents time, we can think of  $d\mathbf{r}/dt$  as the *velocity vector*  $\mathbf{v}$ . If we let  $s$  denote the arclength along  $C$ , measured from some fixed starting point in the positive direction, then in terms of  $s$  the magnitude and direction of  $\mathbf{v}$  are given by

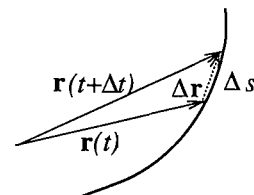
$$(4) \quad |\mathbf{v}| = \left| \frac{ds}{dt} \right|, \quad \text{dir } \mathbf{v} = \begin{cases} \mathbf{t}, & \text{if } ds/dt > 0; \\ -\mathbf{t}, & \text{if } ds/dt < 0. \end{cases}$$

Here  $\mathbf{t}$  is the unit tangent vector (pointing in the positive direction on  $C$  :

$$(5) \quad \mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} .$$

You can see from the picture that  $\mathbf{t}$  is a unit vector, since

$$\left| \frac{d\mathbf{r}}{ds} \right| = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta \mathbf{r}}{\Delta s} \right| = 1 .$$



**2. Line integrals in space.** Let  $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  be a vector field in space, assumed continuous.

We define the *line integral of the tangential component of  $\mathbf{F}$*  along an oriented curve  $C$  in space in the same way as for the plane. We approximate  $C$  by an inscribed sequence of directed line segments  $\Delta\mathbf{r}_k$ , form the approximating sum, then pass to the limit:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{k \rightarrow \infty} \sum_k \mathbf{r}_k \cdot \Delta\mathbf{r}_k .$$

The line integral is calculated just like the one in two dimensions:

$$(6) \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt ,$$

if  $C$  is given by the position vector function  $\mathbf{r}(t)$ ,  $t_0 \leq t \leq t_1$ . Using  $x, y, z$ -components, one would write (6) as

$$(6') \quad \int_C M dx + N dy + P dz = \int_{t_0}^{t_1} \left( M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$$

In particular, if the parameter is the arclength  $s$ , then (6) becomes (since  $\mathbf{t} = d\mathbf{r}/ds$ )

$$(7) \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{s_0}^{s_1} \mathbf{F} \cdot \mathbf{t} ds ,$$

which shows that the line integral is the integral along  $C$  of the tangential component of  $\mathbf{F}$ . As in two dimensions, this line integral represents the work done by the field  $\mathbf{F}$  carrying a unit point mass (or charge) along the curve  $C$ .

**Example 1.** Find the work done by the electrostatic force field  $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$  in carrying a positive unit point charge from  $(1, 1, 1)$  to  $(2, 4, 8)$  along

- a) a line segment    b) the twisted cubic curve  $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ .

**Solution.** a) The line segment is given parametrically by

$$x - 1 = t, \quad y - 1 = 3t, \quad z - 1 = 7t, \quad 0 \leq t \leq 1 .$$

$$\begin{aligned} \int_C y dx + z dy + x dz &= \int_0^1 (3t + 1) dt + (7t + 1) \cdot 3 dt + (t + 1) \cdot 7 dt, \quad \text{using (6')} \\ &= \int_0^1 (31t + 11) dt = \left. \frac{31}{2}t^2 + 11t \right|_0^1 = \frac{31}{2} + 11 = 26.5 . \end{aligned}$$

b) Here the curve is given by  $x = t$ ,  $y = t^2$ ,  $z = t^3$ ,  $1 \leq t \leq 2$ . For this curve, the line integral is

$$\begin{aligned} \int_1^2 t^2 dt + t^3 \cdot 2t dt + t \cdot 3t^2 dt &= \int_1^2 (t^2 + 3t^3 + 2t^4) dt \\ &= \left. \frac{t^3}{3} + \frac{3t^4}{4} + \frac{2t^5}{5} \right|_1^2 \approx 25.18 . \end{aligned}$$

The different results for the two paths shows that for this field, the line integral between two points depends on the path.

### 3. Gradient fields and path-independence.

The two-dimensional theory developed for line integrals in the plane generalizes easily to three-space. For the part where no new ideas are involved, we will be brief, just stating the results, and in places sketching the proofs.

**Definition.** Let  $\mathbf{F}$  be a continuous vector field in a region  $D$  of space. The line integral  $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$  is called **path-independent** if, for any two points  $P$  and  $Q$  in the region  $D$ , the value of  $\int_C \mathbf{F} \cdot d\mathbf{r}$  along a directed curve  $C$  lying in  $D$  and running from  $P$  to  $Q$  depends only on the two endpoints, and not on  $C$ .

An equivalent formulation is (the proof of equivalence is the same as before):

$$(8) \quad \int_P^Q \mathbf{F} \cdot d\mathbf{r} \text{ is path independent} \Leftrightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for every closed curve } C \text{ in } D$$

**Definition** Let  $f(x, y, z)$  be continuously differentiable in a region  $D$ . The vector field

$$(9) \quad \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

is called the **gradient field** of  $f$  in  $D$ . Any field of the form  $\nabla f$  is called a gradient field.

**Theorem. First fundamental theorem of calculus for line integrals.** *If  $f(x, y, z)$  is continuously differentiable in a region  $D$ , then for any two points  $P_1, P_2$  lying in  $D$ ,*

$$(10) \quad \int_{P_1}^{P_2} \nabla f \cdot d\mathbf{r} = f(P_2) - f(P_1),$$

where the integral is taken along any curve  $C$  lying in  $D$  and running from  $P_1$  to  $P_2$ . In particular, the line integral is path-independent.

The proof is exactly the same as before — use the chain rule to reduce it to the first fundamental theorem of calculus for functions of one variable.

There is also an analogue of the second fundamental theorem of calculus, the one where we first integrate, then differentiate.

**Theorem. Second fundamental theorem of calculus for line integrals.**

Let  $\mathbf{F}(x, y, z)$  be continuous and  $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$  path-independent in a region  $D$ ; and define

$$(11) \quad f(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \mathbf{F} \cdot d\mathbf{r}; \quad \text{then} \\ \nabla f = \mathbf{F} \quad \text{in } D.$$

Note that since the integral is path-independent, no  $C$  need be specified in (11). The theorem is proved in your book for line integrals in the plane. The proof for line integrals in space is analogous.

Just as before, these two theorems produce the three equivalent statements: in  $D$ ,

$$(12) \quad \mathbf{F} = \nabla f \Leftrightarrow \int_P^Q \mathbf{F} \cdot d\mathbf{r} \text{ path-independent} \Leftrightarrow \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ for any closed } C$$

As in the two-dimensional case, if  $\mathbf{F}$  is thought of as a force field, then the gradient force fields are called *conservative* fields, since the work done going around any closed path is zero (i.e., energy is conserved). If  $\mathbf{F} = \nabla f$ , then  $f$  is called the (mathematical) *potential function* for  $\mathbf{F}$ ; the physical potential function is defined to be  $-f$ .

**Example 2.** Let  $f(x, y, z) = (x + y^2)z$ . Calculate  $\mathbf{F} = \nabla f$ , and find  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is the helix  $x = \cos t, y = \sin t, z = t, 0 \leq t \leq \pi$ .

**Solution.** By differentiating,  $\mathbf{F} = z\mathbf{i} + 2yz\mathbf{j} + (x + y^2)\mathbf{k}$ . The curve  $C$  runs from  $(1, 0, 0)$  to  $(-1, 0, \pi)$ . Therefore by (10),

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (x + y^2)z \Big|_{(1,0,0)}^{(-1,0,\pi)} = -\pi - 0 = -\pi.$$

No direct calculation of the line integral is needed, notice.

### Exercises: Section 6D