5. Triple Integrals

5A. Triple integrals in rectangular and cylindrical coordinates

5A-1 Evaluate: a) $\int_0^2 \int_{-1}^1 \int_0^1 (x+y+z)dx\,dy\,dz$ b) $\int_0^2 \int_0^{\sqrt{y}} \int_0^{xy} 2xy^2z\,dz\,dx\,dy$

5A-2. Follow the three steps in the notes to supply limits for the triple integrals over the following regions of 3-space.

a) The rectangular prism having as its two bases the triangle in the yz-plane cut out by the two axes and the line $y+z=1$, and the corresponding triangle in the plane $x=1$ obtained by adding 1 to the x-coordinate of each point in the first triangle. Supply limits for three different orders of integration:

(i) $\iiint dz\,dy\,dx$ (ii) $\iiint dx\,dz\,dy$ (iii) $\iiint dy\,dx\,dz$

b)* The tetrahedron having its four vertices at the origin, and the points on the three axes where respectively $x=1$, $y=2$, and $z=2$. Use the order $\iiint dz\,dy\,dx$.

c) The quarter of a solid circular cylinder of radius 1 and height 2 lying in the first octant, with its central axis the interval $0 \le y \le 2$ on the y-axis, and base the quarter circle in the xz-plane with center at the origin, radius 1, and lying in the first quadrant. Integrate with respect to y first; use suitable cylindrical coordinates.

d) The region bounded below by the cone $z^2 = x^2+y^2$, and above by the sphere of radius $\sqrt{2}$ and center at the origin. Use cylindrical coordinates.

5A-3 Find the center of mass of the tetrahedron D in the first octant formed by the coordinate planes and the plane $x+y+z=1$. Assume $\delta=1$.

5A-4 A solid right circular cone of height h with 90^0 vertex angle has density at point P numerically equal to the distance from P to the central axis. Choosing the placement of the cone which will give the easiest integral, find

a) its mass b) its center of mass

5A-5 An engine part is a solid S in the shape of an Egyptian-type pyramid having height 2 and a square base with diagonal D of length 2. Inside the engine it rotates about D. Set up (but do not evaluate) an iterated integral giving its moment of inertia about D. Assume $\delta=1$. (Place S so the positive z axis is its central axis.)

5A-6 Using cylindrical coordinates, find the moment of inertia of a solid hemisphere D of radius a about the central axis perpendicular to the base of D. Assume $\delta=1$..

5A-7 The paraboloid $z=x^2+y^2$ is shaped like a wine-glass, and the plane $z=2x$ slices off a finite piece D of the region above the paraboloid (i.e., inside the wine-glass). Find the moment of inertia of D about the z-axis, assuming $\delta=1$.

5B. Triple Integrals in Spherical Coordinates

5B-1 Supply limits for iterated integrals in spherical coordinates $\iiint d\rho\, d\phi\, d\theta$ for each of the following regions. (No integrand is specified; $d\rho\, d\phi\, d\theta$ is given so as to determine the order of integration.)

a) The region of 5A-2d: bounded below by the cone $z^2 = x^2 + y^2$, and above by the sphere of radius $\sqrt{2}$ and center at the origin.

b) The first octant.

c) That part of the sphere of radius 1 and center at $z = 1$ on the z-axis which lies above the plane $z = 1$.

5B-2 Find the center of mass of a hemisphere of radius a, using spherical coordinates. Assume the density $\delta = 1$.

5B-3 A solid D is bounded below by a right circular cone whose generators have length a and make an angle $\pi/6$ with the central axis. It is bounded above by a portion of the sphere of radius a centered at the vertex of the cone. Find its moment of inertia about its central axis, assuming the density δ at a point is numerically equal to the distance of the point from a plane through the vertex perpendicular to the central axis.

5B-4 Find the average distance of a point in a solid sphere of radius a from
a) the center b) a fixed diameter c) a fixed plane through the center

5C. Gravitational Attraction

5C-1.* Find the gravitational attraction of the solid V bounded by a right circular cone of vertex angle 60° and slant height a, surmounted by the cap of a sphere of radius a centered at the vertex of the cone; take the density to be

(a) 1 (b) the distance from the vertex. Ans.: a) $\pi Ga/4$ b) $\pi Ga^2/8$

5C-2. Find the gravitational attraction of the region bounded above by the plane $z = 2$ and below by the cone $z^2 = 4(x^2 + y^2)$, on a unit mass at the origin; take $\delta = 1$.

5C-3. Find the gravitational attraction of a solid sphere of radius 1 on a unit point mass Q on its surface, if the density of the sphere at $P(x, y, z)$ is $|PQ|^{-1/2}$.

5C-4. Find the gravitational attraction of the region which is bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the sphere $x^2 + y^2 + z^2 = 2z$, on a unit mass at the origin. (Take $\delta = 1$.)

5C-5.* Find the gravitational attraction of a solid hemisphere of radius a and density 1 on a unit point mass placed at its pole. Ans: $2\pi Ga(1 - \sqrt{2}/3)$

5C-6.* Let V be a uniform solid sphere of mass M and radius a. Place a unit point mass a distance b from the center of V. Show that the gravitational attraction of V on the point mass is

$$\text{a) } GM/b^2, \text{ if } b \geq a; \qquad \text{b) } GM'/b^2, \text{ if } b \leq a, \text{ where } M' = \frac{b^3}{a^3} M .$$

Part (a) is Newton's theorem, described in the Remark. Part (b) says that the outer portion of the sphere—the spherical shell of inner radius b and outer radius a —exerts no force on the test mass: all of it comes from the inner sphere of radius b, which has total mass $\frac{b^3}{a^3} M$.

5C-7.* Use Problem 6b to show that if we dig a straight hole through the earth, it takes a point mass m a total of $\pi\sqrt{R/g} \approx 42$ minutes to fall from one end to the other, no matter what the length of the hole is.

(Write $\mathbf{F} = m\mathbf{a}$, letting x be the distance from the middle of the hole, and obtain an equation of simple harmonic motion for $x(t)$. Here

$$R = \text{earth's radius}, \qquad M = \text{earth's mass}, \qquad g = GM/R^2 .)$$