

6. Vector Integral Calculus in Space

6A. Vector Fields in Space

6A-1 Describe geometrically the following vector fields: a) $\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho}$ b) $-x\mathbf{i} - z\mathbf{k}$

6A-2 Write down the vector field where each vector runs from (x, y, z) to a point half-way towards the origin.

6A-3 Write down the velocity field \mathbf{F} representing a rotation about the x -axis in the direction given by the right-hand rule (thumb pointing in positive x -direction), and having constant angular velocity ω .

6A-4 Write down the most general vector field all of whose vectors are parallel to the plane $3x - 4y + z = 2$.

6B. Surface Integrals and Flux

6B-1 Without calculating, find the flux of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the sphere of radius a and center at the origin. Take \mathbf{n} pointing outward.

6B-2 Without calculation, find the flux of \mathbf{k} through the infinite cylinder $x^2 + y^2 = 1$. (Take \mathbf{n} pointing outward.)

6B-3 Without calculation, find the flux of \mathbf{i} through that portion of the plane $x + y + z = 1$ lying in the first octant (take \mathbf{n} pointed away from the origin).

6B-4 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{j}$, and S = the half of the sphere $x^2 + y^2 + z^2 = a^2$ for which $y \geq 0$, oriented so that \mathbf{n} points away from the origin.

6B-5 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = z\mathbf{k}$, and S is the surface of Exercise 6B-3 above.

6B-6 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and S is the part of the paraboloid $z = x^2 + y^2$ lying underneath the plane $z = 1$, with \mathbf{n} pointing generally upwards. Explain geometrically why your answer is negative.

6B-7* Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2}$, and S is the surface of Exercise 6B-2.

6B-8 Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{j}$ and S is that portion of the cylinder $x^2 + y^2 = a^2$ between the planes $z = 0$ and $z = h$, and to the right of the xz -plane; \mathbf{n} points outwards.

6B-9* Find the center of gravity of a hemispherical shell of radius a . (Assume the density is 1, and place it so its base is on the xy -plane.)

6B-10* Let S be that portion of the plane $-12x + 4y + 3z = 12$ projecting vertically onto the plane region $(x - 1)^2 + y^2 \leq 4$. Evaluate

a) the area of S b) $\iint_S z \, dS$ c) $\iint_S (x^2 + y^2 + 3z) \, dS$

6B-11* Let S be that portion of the cylinder $x^2 + y^2 = a^2$ bounded below by the xy -plane and above by the cone $z = \sqrt{(x-a)^2 + y^2}$.

a) Find the area of S . Recall that $\sqrt{1 - \cos \theta} = \sqrt{2} \sin(\theta/2)$. (Hint: remember that the upper limit of integration for the z -integral will be a function of θ determined by the intersection of the two surfaces.)

b) Find the moment of inertia of S about the z -axis. There should be nothing to calculate once you've done part (a).

c) Evaluate $\iint_S z^2 dS$.

6B-12 Find the average height above the xy -plane of a point chosen at random on the surface of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

6C. Divergence Theorem

6C-1 Calculate $\operatorname{div} \mathbf{F}$ for each of the following fields

a) $x^2y\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$ b)* $3x^2yz\mathbf{i} + x^3z\mathbf{j} + x^3y\mathbf{k}$ c)* $\sin^3 x\mathbf{i} + 3y \cos^3 x\mathbf{j} + 2x\mathbf{k}$

6C-2 Calculate $\operatorname{div} \mathbf{F}$ if $\mathbf{F} = \rho^n(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, and tell for what value(s) of n we have $\operatorname{div} \mathbf{F} = 0$. (Use $\rho_x = x/\rho$, etc.)

6C-3 Verify the divergence theorem when $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface composed of the upper half of the sphere of radius a and center at the origin, together with the circular disc in the xy -plane centered at the origin and of radius a .

6C-4* Verify the divergence theorem if \mathbf{F} is as in Exercise 3 and S is the surface of the unit cube having diagonally opposite vertices at $(0,0,0)$ and $(1,1,1)$, with three sides in the coordinate planes. (All the surface integrals are easy and do not require any formulas.)

6C-5 By using the divergence theorem, evaluate the surface integral giving the flux of $\mathbf{F} = x\mathbf{i} + z^2\mathbf{j} + y^2\mathbf{k}$ over the tetrahedron with vertices at the origin and the three points on the positive coordinate axes at distance 1 from the origin.

6C-6 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ over the closed surface S formed below by a piece of the cone $z^2 = x^2 + y^2$ and above by a circular disc in the plane $z = 1$; take \mathbf{F} to be the field of Exercise 6B-5; use the divergence theorem.

6C-7 Verify the divergence theorem when S is the closed surface having for its sides a portion of the cylinder $x^2 + y^2 = 1$ and for its top and bottom circular portions of the planes $z = 1$ and $z = 0$; take \mathbf{F} to be

a) $x^2\mathbf{i} + xy\mathbf{j}$ b)* $zy\mathbf{k}$ c)* $x^2\mathbf{i} + xy\mathbf{j} + zy\mathbf{k}$ (use (a) and (b))

6C-8 Suppose $\operatorname{div} \mathbf{F} = 0$ and S_1 and S_2 are the upper and lower hemispheres of the unit sphere centered at the origin. Direct both hemispheres so that the unit normal is "up", i.e., has positive \mathbf{k} -component.

a) Show that $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, and interpret this physically in terms of flux.

b) State a generalization to an arbitrary closed surface S and a field \mathbf{F} such that $\operatorname{div} \mathbf{F} = 0$.

6C-9* Let \mathbf{F} be the vector field for which all vectors are aimed radially away from the origin, with magnitude $1/\rho^2$.

- What is the domain of \mathbf{F} ?
- Show that $\operatorname{div} \mathbf{F} = 0$.

c) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where S is a sphere of radius a centered at the origin.

Does the fact that the answer is not zero contradict the divergence theorem? Explain.

d) Prove using the divergence theorem that $\iint_S \mathbf{F} \cdot d\mathbf{S}$ over a positively oriented closed surface S has the value zero if the surface does not contain the origin, and the value 4π if it does.

(\mathbf{F} is the vector field for the flow arising from a *source of strength* 4π at the origin.)

6C-10 A flow field \mathbf{F} is said to be *incompressible* if $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for all closed surfaces S . Assume that \mathbf{F} is continuously differentiable. Show that

$$\mathbf{F} \text{ is the field of an incompressible flow} \iff \operatorname{div} \mathbf{F} = 0.$$

6C-11 Show that the flux of the position vector $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through a closed surface S is three times the volume contained in that surface.

6D. Line Integrals in Space

6D-1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following fields \mathbf{F} and curves C :

- $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$; C is the twisted cubic curve $x = t$, $y = t^2$, $z = t^3$ running from $(0, 0, 0)$ to $(1, 1, 1)$.
- \mathbf{F} is the field of (a); C is the line running from $(0, 0, 0)$ to $(1, 1, 1)$
- \mathbf{F} is the field of (a); C is the path made up of the succession of line segments running from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$.
- $\mathbf{F} = zx\mathbf{i} + zy\mathbf{j} + x\mathbf{k}$; C is the helix $x = \cos t$, $y = \sin t$, $z = t$, running from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

6D-2 Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any curve C lying on a sphere of radius a centered at the origin.

6D-3* a) Let C be the directed line segment running from P to Q , and let \mathbf{F} be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot PQ$.

b) Let C be a closed space polygon $P_1P_2 \dots P_nP_1$, and let \mathbf{F} be a constant vector field. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (a).)

c) Let C be a closed space curve, \mathbf{F} a constant vector field. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$. (Use part (b).)

6D-4 a) Let $f(x, y, z) = x^2 + y^2 + z^2$; calculate $\mathbf{F} = \nabla f$.

b) Let C be the helix of 6D-1d above, but running from $t = 0$ to $t = 2n\pi$. Calculate the work done by \mathbf{F} moving a unit point mass along C ; use three methods:

(i) directly

(ii) by using the path-independence of the integral to replace C by a simpler path

(iii) by using the first fundamental theorem for line integrals.

6D-5 Let $\mathbf{F} = \nabla f$, where $f(x, y, z) = \sin(xyz)$. What is the maximum value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ over all possible paths C ? Give a path C for which this maximum value is attained.

6D-6* Let $\mathbf{F} = \nabla f$, where $f(x, y, z) = \frac{1}{x + y + z + 1}$. Find the work done by \mathbf{F} carrying a unit point mass from the origin out to ∞ along a ray.
(Take the ray to be $x = at, y = bt, z = ct$.)

6E. Gradient Fields in Space

6E-1 Which of the following differentials are exact? For each one which is, express it in the form df for a suitable function $f(x, y, z)$, using one of the systematic methods.

a) $x^2 dx + y^2 dy + z^2 dz$ b) $y^2 z dx + 2xyz dy + xy^2 dz$

c) $y(6x^2 + z) dx + x(2x^2 + z) dy + xy dz$

6E-2 Find $\text{curl } \mathbf{F}$, if $\mathbf{F} = x^2 y \mathbf{i} + yz \mathbf{j} + xyz^2 \mathbf{k}$.

6E-3 The fields \mathbf{F} below are defined for all x, y, z . For each,

a) show that $\text{curl } \mathbf{F} = \mathbf{0}$;

b) find a potential function $f(x, y, z)$, using either method, or inspection.

(i) $x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ (ii) $(2xy + z) \mathbf{i} + x^2 \mathbf{j} + x \mathbf{k}$ (iii) $yz \mathbf{i} + xz \mathbf{j} + xy \mathbf{k}$

6E-4 Show that if $f(x, y, z)$ and $g(x, y, z)$ are two functions having the same gradient, then $f = g + c$ for some constant c . (Apply the second fundamental theorem to their difference, or use the first fundamental theorem.)

6E-5 For what values of a and b will $\mathbf{F} = yz^2 \mathbf{i} + (xz^2 + ayz) \mathbf{j} + (bxyz + y^2) \mathbf{k}$ be a conservative field? Using these values, find the corresponding potential function $f(x, y, z)$ by one of the systematic methods.

6E-6 a) Define what it means for $M dx + N dy + P dz$ to be an exact differential.

b) Find all values of a, b, c for which

$$(axyz + y^3 z^2) dx + (a/2)x^2 z + 3xy^2 z^2 + byz^3) dy + (3x^2 y + cxy^3 z + 6y^2 z^2) dz$$

will be exact.

c) For those values of a, b, c , express the differential as df for a suitable $f(x, y, z)$.

6F. Stokes' Theorem

6F-1 Verify Stokes' theorem when S is the upper hemisphere of the sphere of radius one centered at the origin and C is its boundary; i.e., calculate both integrals in the theorem and show they are equal. Do this for the vector fields

a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; b) $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

6F-2 Verify Stokes' theorem if $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and S is the portion of the plane $x + y + z = 0$ cut out by the cylinder $x^2 + y^2 = 1$, and C is its boundary (an ellipse).

6F-3 Verify Stokes' theorem when S is the rectangle with vertices at $(0,0,0)$, $(1,1,0)$, $(0,0,1)$, and $(1,1,1)$, and $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.

6F-4* Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, where M, N, P have continuous second partial derivatives.

a) Show by direct calculation that $\operatorname{div}(\operatorname{curl} F) = 0$.

b) Using (a), show that $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = 0$ for any closed surface S .

6F-5 Let S be the surface formed by the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$, together with the circular disc forming its top, oriented so the normal vector points up or out. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Find the flux of $\nabla \times \mathbf{F}$ through S

(a) directly, by calculating two surface integrals;

(b) by using Stokes' theorem.

6G. Topological Questions

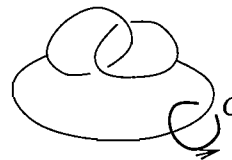
6G-1 Which regions are simply-connected?

- a) first octant b) exterior of a torus c) region between two concentric spheres
 d) three-space with one of the following removed:
 i) a line ii) a point iii) a circle iv) the letter H v) the letter R

6G-2 Show that the fields $\mathbf{F} = \rho^n(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, where $\rho = \sqrt{x^2 + y^2 + z^2}$, are gradient fields for any value of the integer n . (Use $\rho_x = x/\rho$, etc.)

Then, find the potential function $f(x, y, z)$. (It is easiest to phrase the question in terms of differentials: one wants $df = \rho^n(x \, dx + y \, dy + z \, dz)$; for $n = 0$, you can find f by inspection; from this you can guess the answer for $n \neq 0$ as well. The case $n = -2$ is an exception, and must be handled separately. The printed solutions use this method, somewhat more formally phrased using the fundamental theorem of line integrals.)

6G-3* If D is taken to be the exterior of the wire link shown, then the little closed curve C cannot be shrunk to a point without leaving D , i.e., without crossing the link. Nonetheless, show that C is the boundary of a two-sided surface lying entirely inside D . (So if \mathbf{F} is a field in D such that $\operatorname{curl} \mathbf{F} = \mathbf{0}$, the above considerations show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.)



6G-4* In cylindrical coordinates r, θ, z , let $\mathbf{F} = \nabla\varphi$, where $\varphi = \tan^{-1} \frac{z}{r-1}$.

a) Interpret φ geometrically. What is the domain of \mathbf{F} ?

b) From the geometric interpretation what will be the value of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around a closed path C that links with the unit circle in the xy -plane (for example, take C to be the circle in the yz -plane with radius 1 and center at $(0, 1, 0)$?)

6H. Applications to Physics

6H-1 Prove that $\nabla \cdot \nabla \times \mathbf{F} = 0$. What are the appropriate hypotheses about the field \mathbf{F} ?

6H-2 Show that for any closed surface S , and continuously differentiable vector field \mathbf{F} ,

$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0.$$

Do it two ways: a) using the divergence theorem; b) using Stokes' theorem.

6H-3* Prove each of the following (ϕ is a (scalar) function):

a) $\nabla \cdot (\phi \mathbf{F}) = \phi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \phi$

b) $\nabla \times (\phi \mathbf{F}) = \phi \nabla \times \mathbf{F} + (\nabla \phi) \times \mathbf{F}$

c) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

6H-4* The *normal derivative*. If S is an oriented surface with unit normal vector \mathbf{n} , and ϕ is a function defined and differentiable on some domain containing S , then the **normal derivative** of ϕ on S is defined to be the directional derivative of ϕ in the direction \mathbf{n} . In symbols (on the left is the notation for the normal derivative):

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \mathbf{n}.$$

Prove that if S is closed and D its interior, and if ϕ has continuous second derivatives inside D , then

$$\iint_S \frac{\partial \phi}{\partial n} dS = \iiint_D \nabla^2 \phi dV.$$

(This shows for example that if you are trying to find a *harmonic* function ϕ defined in D and having a prescribed normal derivative on S , you must be sure that $\frac{\partial \phi}{\partial n}$ has been prescribed so that $\iint_S \frac{\partial \phi}{\partial n} dS = 0$.)

6H-5* Formulate and prove the analogue of the preceding exercise for the plane.

6H-6* Prove that, if S is a closed surface with interior D , and ϕ has continuous second derivatives in D , then

$$\iint_S \phi \frac{\partial \phi}{\partial n} dS = \iiint_D \phi (\nabla^2 \phi) + (\nabla \phi)^2 dV.$$

6H-7* Formulate and prove the analogue of the preceding exercise for a plane.

6H-8 A boundary value problem.* Suppose you want to find a function ϕ defined in a domain containing a closed surface S and its interior D , such that (i) ϕ is harmonic in D and (ii) $\phi = 0$ on S .

a) Show that the two conditions imply that $\phi = 0$ on all of D . (Use Exercise 6.)

b) Instead of assuming (ii), assume instead that the values of ϕ on S are prescribed as some continuous function on S . Prove that if a function ϕ exists which is harmonic in D and has these prescribed boundary values, then it is unique — there is only one such function. (In other words, the values of a harmonic function on the boundary surface S determine its values everywhere inside S .) (Hint: Assume there are two such functions and consider their difference.)

6H-9 Vector potential* In the same way that $\mathbf{F} = \nabla\phi \Rightarrow \nabla \times \mathbf{F} = \mathbf{0}$ has the partial converse

$$\nabla \times \mathbf{F} = \mathbf{0} \text{ in a simply-connected region} \Rightarrow \mathbf{F} = \nabla f,$$

so the theorem $\mathbf{F} = \nabla \times \mathbf{G} \Rightarrow \nabla \cdot \mathbf{F} = 0$ has the partial converse

$$(*) \quad \nabla \cdot \mathbf{F} = 0 \text{ in a suitable region} \Rightarrow \mathbf{F} = \nabla \times \mathbf{G}, \text{ for some } \mathbf{G}.$$

\mathbf{G} is called a **vector potential** for \mathbf{F} . A suitable region is one with this property: whenever P lies in the region, the whole line segment joining P to the origin lies in the region. (Instead of the origin, one could use some other fixed point.) For instance, a sphere, a cube, or all of 3-space would be suitable regions.

Suppose for instance that $\nabla \cdot \mathbf{F} = 0$ in all of 3-space. Then \mathbf{G} exists in all of 3-space, and is given by the formula

$$(**) \quad \mathbf{G} = \int_0^1 t \mathbf{F}(tx, ty, tz) \times \mathbf{R} dt, \quad \mathbf{R} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$$

The integral means: integrate separately each component of the vector function occurring in the integrand, and you'll get the corresponding component of \mathbf{G} .

We shall not prove this formula here; the proof depends on Leibniz' rule for differentiating under an integral sign. We can however try out the formula.

a) Let $\mathbf{F} = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$. Check that $\text{div } \mathbf{F} = 0$, find \mathbf{G} from the formula (**), and check your answer by verifying that $\mathbf{F} = \text{curl } \mathbf{G}$.

b) Show that \mathbf{G} is unique up to the addition of an arbitrary gradient field; i.e., if \mathbf{G} is one such field, then all others are of the form

$$(***) \quad \mathbf{G}' = \mathbf{G} + \nabla f,$$

for an arbitrary function $f(x, y, z)$. (Show that if \mathbf{G}' has the form (***), then $\mathbf{F} = \text{curl } \mathbf{G}'$; then show conversely that if \mathbf{G}' is a field such that $\text{curl } \mathbf{G}' = \mathbf{F}$, then \mathbf{G}' has the form (***)).

6H-10 Let \mathbf{B} be a magnetic field produced by a moving electric field \mathbf{E} . Assume there are no charges in the region. Then one of Maxwell's equations in differential form reads

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}.$$

What is the integrated form of this law? Prove your answer, as in the notes; you can assume that the partial differentiation can be moved outside of the integral sign.

6H-11* In the preceding problem if we also allow for a field \mathbf{j} which gives the current density at each point of space, we get Ampere's law in differential form (as modified by Maxwell):

$$\nabla \times \mathbf{B} = \frac{1}{c} \left(4\pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \right).$$

Give the integrated form of this law, and deduce it from the differential form, as done in the notes.