MODEL ANSWERS TO HWK #12
(18.022 FALL 2010)

(1) (i) $\nabla \times F = (2by - 2y)\hat{\imath} + (2y - 2ay)\hat{k} = 0$. Hence $a = b = 1$.
(ii) $f_x = y^2$, so $f = xy^2 + g(y,z)$. $f_y = 2xy + g_y = 2xy + 2yz$, so $g = y^2z + h(z)$. Now, $f_z = y^2 + h' = y^2 + z^2$, and $h = \frac{z^3}{3}$. Therefore $f = xy^2 + y^2z + \frac{z^3}{3}$.
(iii) For conservative F, $\int_C F \cdot d\mathbf{s} = f(b) - f(a)$ for the end points a and b of C. So the surface defined by $f(x,y,z) = c$ for some constant c will do. Therefore $xy^2 + y^2z + \frac{z^3}{3} = c$ for some constant.

(2) Parameterize the surface by $\mathbf{X}(x,y) = (x,y,y)$, where the range of x and y are the rectangle $[0,1] \times [0,2]$. Then $X_x \times X_y = (0,-1,1)$. So $\iint_S F \cdot d\mathbf{S} = \int_0^2 \int_0^1 x^2 + y^2 dx dy = \frac{10}{3}$.

(3) F is smooth everywhere except those three points. By Green's theorem, $\oint_{C_2(P_0)} F \cdot d\mathbf{s} + \oint_{C_1(P_1)} F \cdot d\mathbf{s} = \oint_{C_6(P_0)} F \cdot d\mathbf{s}$, hence $\oint_{C_1(P_1)} F \cdot d\mathbf{s} = 1 - (-2) = 3$. Similarly, since $\oint_{C_6(P_0)} F \cdot d\mathbf{s} + \oint_{C_1(P_2)} F \cdot d\mathbf{s} = \oint_{C_{10}(P_0)} F \cdot d\mathbf{s}$, hence $\oint_{C_1(P_2)} F \cdot d\mathbf{s} = 3 - 1 = 2$. Now, $\oint_{C_6(P_2)} F \cdot d\mathbf{s} = \oint_{C_1(P_1)} F \cdot d\mathbf{s} + \oint_{C_1(P_2)} F \cdot d\mathbf{s}$, and we get $\oint_{C_6(P_2)} F \cdot d\mathbf{s} = 3 + 2 = 5$.

(4) (6.3.16) $\nabla \times F = 0$ gives us $6xy\sin(xz) + 5 = -axy\sin(xz) + b$, $-ayz\sin(xz) = 6yz\sin(xz)$. Hence $a = -6, b = 5$.

(5) (7.1.4)
(a) $X_s \times X_t = (-s^2\cos t, -s^2\sin t, 2s^3)$. Hence, (-1,0,-2).
(b) By (a), $-(x-1) - 2(z+1) = 0$, or $x + 2z = -1$.
(c) $x^2 + y^2 - z^4 = 0$.

(6) (7.1.20) The normal vector field is $\mathbf{N}(s,t) = \begin{vmatrix} \lambda\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 1 \end{vmatrix} = (\sin\theta, -\cos\theta, r)$. The surface area will be

$$\int_0^{2\pi n} \int_0^1 \sqrt{\sin^2\theta + \cos^2\theta + r^2} dr d\theta = 2\pi n \int_0^1 \sqrt{1 + r^2} dr = 2\pi n \int_0^{\text{arcsinh}(1)} \cosh^2 t \, dt$$
$$= \pi n \int_0^{\text{arcsinh}(1)} (1 + \cosh 2t) dt = \pi n(\text{arcsinh}1 + \sqrt{2}).$$

(7) (7.2.13)

$$\iint_S x^2 dS = \frac{1}{2} \iint_S (x^2 + y^2) dS = \frac{1}{2} \left(\iint_{\text{bottom}} r^2 dS + \iint_{\text{top}} r^2 dS + \iint_{\text{side}} 9\, dS \right)$$
$$= \int_0^{2\pi} \int_0^3 r^2 r \, dr d\theta + \frac{9}{2} \iint_{\text{side}} dS = 2\pi \left(\frac{r^4}{4} \bigg|_{r=0}^3 + \frac{9}{2} 2\pi \cdot 3 \cdot 4 = \frac{297}{2}\pi \right.$$

(8) (7.2.17) The unit normal vectors to the top ($\mathbf{k}$), bottom ($-\mathbf{k}$), and side ($\frac{1}{3}(x\mathbf{i} + y\mathbf{j})$) surfaces of the cylinder are perpendicular to the vector field $\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$ being integrated, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$.

(9) (7.3.11) The boundary of S is the circle $y = 1, x^2 + z^2 = 9$, which also bounds the flat disc $y = 1, x^2 + z^2 \leq 9$. For this disc, the rightward-pointing normal is $\mathbf{j}$, so we only need to calculate the second component of $\nabla \times \mathbf{F}$, which will be 5.

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D 5\, dS = 5\pi 3^3 = 45\pi.$$

(10) (7.3.13)

(a) $\sin(2t) = 2(\cos t)(\sin t)$, so $\mathbf{x}(t) = (\cos t, \sin t, \sin(2t))$ lies on the surface $z = 2xy$.

(b) The closed curve above is the boundary of the surface $z = 2xy, x^2 + y^2 \leq 1$, which in turn is parametrized by $\mathbf{X}(r,t) = (r\cos t, r\sin t, 2r^2 \cos t \sin t)$, with $0 \leq t \leq 2\pi$ and $0 \leq r \leq 1$. The normal vector field is $\mathbf{N}(r,t) = \frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial t} = (-2r^2 \sin t, -2r^2 \cos t, r)$. Also, the curl of the vector field $\mathbf{F}(x,y,z) = (y^3 + \cos x, \sin y + z^2, x)$ is $\nabla \times \mathbf{F} = (-2z, -1, -3y^2)$. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} =$$

$$= \int_0^{2\pi} \int_0^1 (-4r^2 \cos t \sin t, -1, -3r^2 \sin^2 t) \cdot (-2r^2 \sin t, -2r^2 \cos t, r) dr\, dt = \ldots = -\frac{3\pi}{4}.$$

(11) (7.3.16) Let D be the solid unit cube and B its bottom square. Then by Gauss' theorem, $\iiint_D \nabla \cdot \mathbf{F} dV = \iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot d\mathbf{S} + \iint_B \mathbf{F} \cdot d\mathbf{S}$. The we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D (2xz\mathrm{e}^{x^2} + 3 - 7yz^6) dV - \iint_B \mathbf{F} \cdot (-\mathbf{k}) dS$$

$$= \int_0^1 2x\mathrm{e}^{x^2} dx \int_0^1 z\, dz + 3 - \int_0^1 y\, dy \int_0^1 7z^6 dz + \int_0^1 \int_0^1 2dxdy = 4 + \frac{\mathrm{e}}{2}.$$

(12) (7.3.18)

(a) The boundary of D is the union of S_7 (with normal pointed outward) and S_5 (with normal pointad inward):

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_{S_7} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_5} \mathbf{F} \cdot d\mathbf{S} = 7a + b - 5a - b = 2a.$$

(b) If $\mathbf{F} = \nabla \times \mathbf{G}$, we use Gauss' theorem followed by Stokes' theorem. Note that ∂D is already a surface without boundary, so $\partial(\partial D)$ is the empty set:

$$\iiint_D \nabla \cdot \nabla \times \mathbf{G}\, dV = \iint_{\partial D} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial(\partial D)} \mathbf{F} \cdot d\mathbf{s} = 0.$$

(13) (7.3.19)

(a) At points of S, we have

$$\frac{\partial f}{\partial n} = \nabla f \cdot \mathbf{n} = (\frac{2x}{a^2}, \frac{2y}{a^2}, \frac{2z}{a^2}) \cdot (\frac{x}{a}, \frac{y}{a}, \frac{z}{a}) = \frac{2(x^2 + y^2 + z^2)}{a^3} = \frac{2}{a},$$

so

$$\iint_S \frac{\partial f}{\partial n} dS = \iint_S \frac{2}{a} dS = \frac{2}{a}\frac{1}{8} 4\pi a^2 = \pi a.$$

(b) We have $\nabla \cdot (\nabla f) = \nabla \cdot (\frac{2x}{\rho^2}, \frac{2y}{\rho^2}, \frac{2z}{\rho^2}) = \ldots = \frac{2}{\rho^2}$, so

$$\iiint_D \nabla \cdot (\nabla f)\, dV = 2 \iiint_D \frac{1}{\rho^2}\, dV = 2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^a \frac{1}{\rho^2}\, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \pi a.$$

(c) The three flat quarter circles that are part of ∂D do not contribute anything to $\iint_S \nabla f \cdot \mathbf{n}\, dS$. For example, on the bottom quarter circle, $\nabla f(x, y, 0) = (\frac{2x}{\rho^2}, \frac{2y}{\rho^2}, 0)$ and the unit normal is $-\mathbf{k}$, so $\iint_{\text{bottom}} \nabla f \cdot (-\mathbf{k}) dS = 0$. The cases of the other two are similar.

WE HOPE YOU ENJOYED 18.022,

AND GOOD LUCK ON THE FINAL!!!