Solutions for PSet 5

1. (8.14:10)

   (a) By hypothesis, we know $\bigtriangledown f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in B(\mathbf{a})$. By definition $\bigtriangledown f(\mathbf{x}) = (D_1 f(\mathbf{x}), \ldots, D_n f(\mathbf{x}))$, thus $D_1 f(\mathbf{x}) = \cdots = D_n f(\mathbf{x}) = 0$. By the 1-dimensional theorem, this means that $f(\mathbf{x})$ is constant on every line $\mathbf{x} + t\mathbf{e}_1, \ldots \mathbf{x} + t\mathbf{e}_n$ for any $\mathbf{x} \in B(\mathbf{a})$. Let $f(\mathbf{a}) = c$. Given $\mathbf{y} \in B(\mathbf{a})$, we will prove that $f(\mathbf{y}) = c$. Let $\mathbf{y} - \mathbf{a} = d_1\mathbf{e_1} + \cdots + d_n\mathbf{e_n}$, then:

   $$c = f(\mathbf{a}) = f(\mathbf{a} + d_1\mathbf{e}_1) = \cdots = f(\mathbf{a} + d_1\mathbf{e_1} + \cdots + d_n\mathbf{e_n}) = f(\mathbf{y}).$$

   (b) The condition means, that $\mathbf{a}$ is a local maximum of the function f. In particular, because $f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in B(\mathbf{a})$, 0 is a maximum for all the functions $f(\mathbf{a} + t\mathbf{e}_i)$. By the 1-dimensional theorem $D_i(f(\mathbf{a})) = f'(\mathbf{a} + t\mathbf{e}_i) = 0$, and thus

   $$\bigtriangledown f(\mathbf{a}) = (D_1 f(\mathbf{a}), \ldots, D_n f(\mathbf{a})) = \mathbf{0}.$$

2. (8.17:6) $f(x,y) = \sqrt{|xy|} = \sqrt{|x|}\sqrt{|y|}$

   (a) $f(x,y) = 0$ on the lines $(x,0)$ and $(0,y)$. Thus $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at the origin.

   (b) For f to have a tangent plane at the origin, it must have a total derivative at the origin. If indeed f had a total derivative at the origin then we expect

   $$f'((x,y);(1,1)) \text{ to be } (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \cdot (1,1) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0 + 0 = 0.$$

   But in reality, $f'((x, y = x);(1,1))$ is the partial derivative along the line $x = y$. On this line, $f(x,x) = x$, thus $f'((x,y);(1,1)) = 1$. This defies our expectation. Thus f cannot have a total derivative, nor a tangent plane at the origin.

3. (8.17:10) Denote $f(x,y,z) = (x-c)^2 + y^2 + z^2$ and $g(x,y,z) = x^2 + (y-1)^2 + z^2$. Then the spheres in question are given by:

   $$\begin{aligned} L_f(3) &= \{(x,y,z) : f(x,y,z) = (x-c)^2 + y^2 + z^2 = 3\} \\ L_g(1) &= \{(x,y,z) : g(x,y,z) = x^2 + (y-1)^2 + z^2 = 1\} \end{aligned}$$

At a point (x, y, z) where the two spheres intersect:

$$\begin{aligned} \text{gradient of } L_f(3) \text{ is } \ \bigtriangledown f(x,y,z) &= (2(x-c), 2y, 2z) \\ \text{gradient of } L_g(1) \text{ is } \ \bigtriangledown g(x,y,z) &= (2x, 2(y-1), 2z) \end{aligned}$$

and $T(L_f(3))$, $T(L_g(1))$ are the respective tangent planes of $L_f(3)$, $L_g(1)$.

By definition, the gradient is perpendicular to the tangent plane. Or $\bigtriangledown f(x,y,z)$ is perpendicular to $T(L_f(3))$ and $\bigtriangledown g(x,y,z)$ perpendicular to $T(L_g(1))$. Thus the tangent planes $T(L_f(3))$ and $T(L_g(1))$ are perpendicular at the intersection if and only if the gradients are perpendicular to each other at the intersection:

$$\bigtriangledown f(x,y,z) \cdot \bigtriangledown g(x,y,z) = 0 \ \text{ or } \ 4x(x-c) + 4y(y-1) + 4z^2 = 0$$

To find c for the above condition to hold, we can solve the system of 3 equations:

$$\begin{aligned} (x-c)^2 + y^2 + z^2 &= 3 \\ x^2 + (y-1)^2 + z^2 &= 1 \\ 4x(x-c) + 4y(y-1) + 4z^2 &= 0 \end{aligned}$$

Add the first two equations to get $2(x^2+y^2+z^2) - 2xc - 2y = 3 - c^2$. This, combined with the last equation, gives $c^2 = 3$. Thus $c = \pm\sqrt{3}$.

4. Consider $\mathbf{a} \in \mathbb{R}^n$. To prove continuity at $\mathbf{a}$ we need to find a $\delta > 0$ for every given $\epsilon > 0$ with the property that $|\mathbf{x} - \mathbf{a}| < \delta$ implies $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$. Note that the set of points $f(\mathbf{x})$ for which $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ are contained in an open ball $B_\epsilon(f(\mathbf{a}))$. Now the preimage of this open ball $f^{-1}(B_\epsilon(f(\mathbf{a})))$ is open but may not be connected. That is, it may consist of the union of a few disjoint open sets. But, necessarily, one of these open sets contains $\mathbf{a}$. Call this open set U. Then openness implies there exists $\delta > 0$ such that $B_\delta(\mathbf{a}) \subset U$. Moreover, as $f(U) \subset B_\epsilon(f(\mathbf{a}))$, containment implies $f(B_\delta(\mathbf{a})) \subset B_\epsilon(f(\mathbf{a}))$. It follows that f is continuous.

5. $T_{\mathbf{a}}$ is defined such that:

$$f(\mathbf{x}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{x} - \mathbf{a}) + |\mathbf{x} - \mathbf{a}| E(\mathbf{a}, \mathbf{x} - \mathbf{a})$$

where $E(\mathbf{a}, \mathbf{x} - \mathbf{a}) \to 0$ as $|\mathbf{x} - \mathbf{a}| \to 0$.

Thus $T_{\mathbf{a}} = f$ if and only if:

$$f(\mathbf{x}) = f(\mathbf{a}) + f(\mathbf{x} - \mathbf{a}) + |\mathbf{x} - \mathbf{a}| E(\mathbf{a}, \mathbf{x} - \mathbf{a})$$

where $E \to 0$ as $|\mathbf{x} - \mathbf{a}| \to 0$.

Since f is a linear transformation,

$$f(\mathbf{x}) = f(\mathbf{a}) + f(\mathbf{x} - \mathbf{a})$$

and thus $E(\mathbf{a}, \mathbf{x} - \mathbf{a}) \equiv 0$ as $|\mathbf{x} - \mathbf{a}| \to 0$.