18.02 Problem Set 3

At MIT problem sets are referred to as 'psets'. You will see this term used occasionally within the problems sets.

The 18.02 psets are split into two parts 'part I' and 'part II'. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website. The intention is that these help the student develop some fluency with concepts and techniques. Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

Part I (10 points)

At MIT the underlined problems must be done and turned in for grading.
The 'Others' are *some* suggested choices for more practice.

A listing like '§1B : $\underline{2}$, 5$\underline{\text{b}}$, $\underline{10}$' means do the indicated problems from supplementary problems section 1B.

1 Parametric equations for lines and curves
§1E: 3$\underline{\text{c}}$, $\underline{4}$; Others: 2, 3ab, 5, 7
§1I: 3$\underline{\text{ab}}$, $\underline{5}$; Others: 1, 3d, 7

2 Vector calculus of curves. Kepler's second law
§1J: $\underline{2}$, $\underline{4}$, $\underline{6}$, $\underline{9}$; Others: 1, 3, 5
§1K: $\underline{3}$

Part II (15 20)

Problem 1 (5: 1,3,1)
A circular disk of radius 2 has a dot marked at a point half-way between the center and the circumference. Denote this point by P. Suppose that the disk is tangent to the x-axis with the center initially at $(0, 2)$ and P intially at $(0, 1)$, and that it starts to roll to the right on the x-axis at unit speed. Let C be the curve traced out by the point P.

a) Make a sketch of what you think the curve C will look like.

Note: any sketch (except a unicorn with zebra stripes) gets the 1 point of credit.

b) Use vectors to find the parametric equations for $\overrightarrow{OP}$ as a function of time t.

c) Open the 'Mathlet' **Wheel** (with link on course webpage) and set the parameters to view an animation of this particular motion problem. Then activate the 'Trace' function to see a graph of the curve C. If this graph is substantially different from your hand sketch, sketch it also and then describe what led you to produce your first idea of the graph. (The mathlet is right, by the way; and *No Fair Working Backwards* from the mathlet – the object of the exercise is to give it a try first.)

Problem 2 (5: 2,2,1)
a) Let $\mathbf{u}$ and $\mathbf{v}$ be two non-parallel unit vectors with $\mathbf{u} \perp \mathbf{v}$, and let
$\mathbf{r}(t) = \mathbf{u}\cos(t) + \mathbf{v}\sin(t)$.
Show that the curve $\mathbf{r}(t)$ sweeps out the unit circle centered at O in the plane $\mathcal{P}$ defined by $\mathbf{u}$ and $\mathbf{v}$ (i.e. the plane through the origin which contains $\mathbf{u}$ and $\mathbf{v}$).

b) Use the result of part(a) to find the parametric equations of C = the circle of radius 1 centered at the origin which lies in the plane $\mathcal{P}: \; x + 2y + z = 0$

c) Give a rough sketch of C lying in $\mathcal{P}$.

Problem 3 (5: 3,2)
a) Find the equations of all the lines which pass through the origin and which lie in the plane $\mathcal{P}$ given by the equation $x + 2y + z = 0$.

b) Illustrate with a (*rough*) sketch, and then give a geometric description of this family of lines.

(Suggestion: what is the smallest number of variables needed to describe the family, and how does that fact relate to what you see on the sketch?)

Problem 4 (5: 1,3,1). A model for a photo enlarger.
A simple mathematical model of a way to enlarge a plane figure is to put the transparent plane containing the figure in a horizontal position, place a point light source at some distance above the plane, and then project the figure – i.e. its shadow – onto a parallel plane at some distance on the other side from the light. In this problem we'll use vector methods to compute the distortion created when the two planes are slightly out of parallel, for the case of a simple figure.

Suppose that the light source is at the point (0, 0, 4) and that the figure to be projected is the cicle $\mathcal{C} = \quad x = \cos t, \quad y = \sin t, \quad z = 2$ in the horizontal plane $z = 2$. The imaging plane is meant to be the plane $z = 0$, in order to produce an enlarged circle. Suppose instead, however, that the bottom plane is slghtly tilted. We'll take this tilted plane $\mathcal{P}_\alpha$ to be given by the equation $m\,y + z = 0$, with $m = \tan\alpha$. Check first that this is the plane with normal $\langle 0, \sin\alpha, \cos\alpha\rangle$, so that $\mathcal{P}_\alpha$ contains the x-axis and is tilted to the x-y plane with angle $-\alpha$ (if $\alpha > 0$). The horizontal plane $z = 2$ and $\mathcal{P}_\alpha$ are thus slightly out of parallel if $\alpha \approx 0$.

a) Make a sketch showing the situation described above.

b) Show that the equation of the curve $\mathcal{C}_\alpha :=$ the shadow (or projection) of the curve $\mathcal{C}$ in the plane $\mathcal{P}_\alpha$ is given in vector-parametric form by

$$\mathbf{r}_\alpha(t) = \left(\frac{4\cos t}{2 - m\sin t}\right)\mathbf{i} + \left(\frac{4\sin t}{2 - m\sin t}\right)\mathbf{j} + \left(\frac{-4\,m\sin t}{2 - m\sin t}\right)\mathbf{k}.$$

c) Check that when $\alpha = 0$ the curve $\mathbf{r}_0(t)$ is the enlarged circle $\mathcal{C}_0$ in the x-y plane. Then use the following 'quick-and-dirty' method to estimate the distortion in $\mathcal{C}_\alpha$ from the circle $\mathcal{C}_0$ caused by the tilt: compute the distance $|\mathbf{r}_\alpha(t) - \mathbf{r}_0(t)|$ between the two curves at the four 'corner' points corresponding to $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and just take the largest value from these four.