Parametric equations of lines

General parametric equations

In this part of the unit we are going to look at parametric curves. This is simply the idea that a point moving in space traces out a path over time. Thus there are four variables to consider, the position of the point (x, y, z) and an independent variable t, which we can think of as time. (If the point is moving in plane there are only three variables, the position of the point (x, y) and the time t.)

Since the position of the point depends on t we write

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

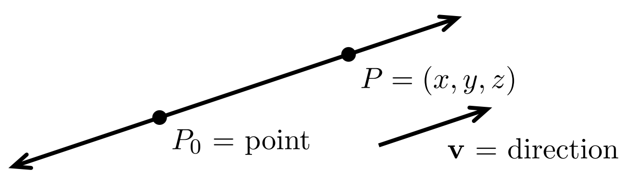
to indicate that x, y and z are functions of t. We call t the parameter and the equations for x, y and z are called *parametric equations.*

In physical examples the parameter often represents time. We will see other cases where the parameter has a different interpretation, or even no interpretation.

Parametric equations of lines

Later we will look at general curves. Right now, let's suppose our point moves on a line.

The basic data we need in order to specify a line are a point on the line and a vector parallel to the line. That is, we need a point and a direction.



Example 1: Write parametric equations for a line through the point $P_0 = (1, 2, 3)$ and parallel to the vector $\mathbf{v} = \langle 1, 3, 5\rangle$.

<u>Answer:</u> If $P = (x, y, z)$ is on the line then the vector

$$\overrightarrow{\mathbf{P_0P}} = \langle x - 1, y - 2, z - 3\rangle$$

is parallel to $\langle 1, 3, 5\rangle$. That is, $\overrightarrow{\mathbf{P_0P}}$ is a scalar multiple of $\langle 1, 3, 5\rangle$. We call the scale t and write:

$$\begin{aligned} & \langle x, y, z\rangle = \langle x - 1, y - 2, z - 3\rangle = t\langle 1, 3, 5\rangle \\ \Leftrightarrow \quad & x - 1 = t, \ \ y - 2 = 3t, \quad z - 3 = 5t \\ \Leftrightarrow \quad & x = 1 + t, \ \ y = 2 + 3t, \quad z = 3 + 5t. \end{aligned}$$

Example 2: In example 1, if our direction vector was $\langle 2, 6, 10\rangle = 2\mathbf{v}$ we would get the same line with a different parametrization. That is, the moving point's trajectory would follow the same path as the trajectory in example 1 , but would arrive at each point on the line at a different time.

Example 3: In general, the line through $P_0 = (x_0, y_0, z_0)$ in the direction of (i.e., parallel to) $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ has parametrization

$$\begin{aligned} & \langle x, y, z\rangle = \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3\rangle \\ \Leftrightarrow \quad & x = x_0 + tv_1, \quad y = y_0 + tv_2 \quad z = z_0 + tv_3. \end{aligned}$$

Example 4: Find the line through the point $P_0 = (1, 2, 3)$ and $P_1 = (2, 5, 8)$.

Answer: We use the data given to find the basic data (a point and direction vector) for the line.

We're given a point, $P_0 = (1, 2, 3)$. The direction vector $\mathbf{v} = \overrightarrow{\mathbf{P_0 P_1}} = \langle 1, 3, 5 \rangle$. So, we get

$$\begin{aligned} & \langle x, y, z \rangle = \overrightarrow{\mathbf{OP_0}} + t\mathbf{v} = \langle 1 + t, 2 + 3t, 3 + 5t \rangle \\ \Leftrightarrow \quad & x = 1 + t, \quad y = 2 + 3t, \quad z = 3 + 5t. \end{aligned}$$