18.02 Problem Set 6

At MIT problem sets are referred to as 'psets'. You will see this term used occasionally within the problems sets.

The 18.02 psets are split into two parts 'part I' and 'part II'. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website. The intention is that these help the student develop some fluency with concepts and techniques. Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

Part I (10 points)

At MIT the underlined problems must be done and turned in for grading.
The 'Others' are *some* suggested choices for more practice.

A listing like '§1B : $\underline{2}$, 5$\underline{\text{b}}$, $\underline{10}$' means do the indicated problems from supplementary problems section 1B.

1 Lagrange multipliers.
§2I: $\underline{1}$, $\underline{3}$; Others: 2, 4

2 Non-independent variables.
§2J: $\underline{1}$, $\underline{2}$, $\underline{3}$a, 4$\underline{\text{b}}$, 5$\underline{\text{a}}$, $\underline{7}$; Others: 3b, 4a, 6

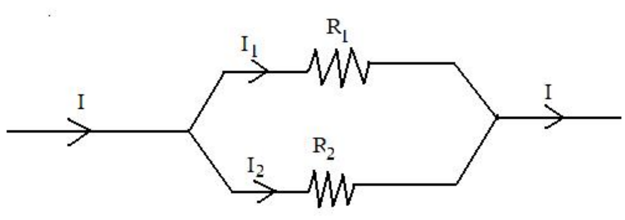
Part II (15 points)

Problem 1 (4: 2,1,1)
Go to the 'Mathlet' **Lagrange Multipliers** (with link on the course webpage), and choose $f(x,y) = x^2 - y^2 \quad g(x,y) = x^2 + y^2$.
a) Solve by hand to find the two values of λ and the possiblities for the corresponding points (x,y) at which the gradients are proportional. Then check these possibities on the applet and verify the predicted proportionality on the graph.
b) Now take $b = 3$ and finish the solution of part(a) by hand to find the possible points which may give a relative extremum of f. Then return to the applet, set $b = 3$, move the f-levels until they makes contact with the $g = 3$ constraint curve, and read the values of f at the points of contact. Compare with the results found by hand; how close could you get?
c) What do the two values of λ correspond to in terms of the pairs of solution points? Do the gradients of f and g point in the same or the opposite direction at the contact points in the two different cases, and is this consistent with the signs of λ?

Problem 2 (4: 2,2)
In this problem we examine how electricity flows through circuits to minimize energy.

A current I flowing over a resistor R results in an energy loss (in the form of heat/light) equal to I^2R per second. It turns out that, in a sense, "electricity prefers to flow in the way that minimizes energy loss to resistance". For example, when an electric current comes to a fork, it will divide itself up in such a way that a large portion of the current flows where the resistance is low and a small portion flows where the resistance is high (you might think all the electricity would flow where the resistance is low but the energy loss is proportional to I^2 so it is better to spread the current around).

Suppose we have the following situation where a current I comes to a pair of resistors in parallel:



a) Determine what choice of I_1 and I_2 will minimize energy loss and hence determine what the currents will be along the two paths. (Alternatively, if you already are familiar with resistors in parallel and current flows, verify that the currents I_1 and I_2 do in fact minimize energy loss).

b) Suppose instead we had three resistors in parallel. In terms of R_1,R_2, and R_3 determine the values of I_1, I_2, and I_3 which minimize energy loss.

Problem 3 (7: 1,2,2,2)
Using the usual rectangular and polar coordinates, let w be the area of the right triangle in the first quadrant having its vertices at $(0,0)$, $(x,0)$ and (x,y). Using the equation expressing w in terms of x and y and the equations expressing y in terms of x and θ, calculate the two partial derivatives $\left(\frac{\partial w}{\partial x}\right)_\theta$ and $\left(\frac{\partial w}{\partial \theta}\right)_x$ in three different ways.

a) Directly, by first expressing w in terms of the independent variables x and θ.

b) By using the chain rule – for example $\left(\frac{\partial w}{\partial x}\right)_\theta = w_x \left(\frac{\partial x}{\partial x}\right)_\theta + w_y \left(\frac{\partial y}{\partial x}\right)_\theta$, where w_x and w_y are the formal partial derivatives.

c) By using differentials.

d) Using the triangle picture and geometric intuition, estimate $\left(\frac{\Delta w}{\Delta x}\right)_\theta$ and $\left(\frac{\Delta w}{\Delta \theta}\right)_x$ and show they agree with the two corresponding partial derivatives.