Problems: Two Dimensional Curl

Imagine a flat arrangement of particles covering the plane. Suppose all the particles are moving in counterclockwise circles about the origin with constant angular speed ω.

Let $\mathbf{F}(x, y)$ be the velocity field described by the velocity of the particles at point (x, y). Find $\mathbf{F}$ and show $\text{curl}(\mathbf{F}) = 2\omega$.

Answer: Because the particles have a constant angular speed ω and no radial velocity, the motion of the particles can be parametrized by $r = r_0$, $\theta = \theta_0 + \omega t$. In polar coordinates we have $(x(t), y(t)) = (r_0 \cos(\theta_0 + \omega t), r_0 \sin(\theta_0 + \omega t))$.

Taking derivatives with respect to t we find

$$\begin{aligned}
\mathbf{F} &= -\omega\, r_0 \sin(\theta_0 + \omega t)\mathbf{i} + \omega\, r_0 \cos(\theta_0 + \omega t)\mathbf{j} = \langle -\omega y, \omega x \rangle, \\
\text{curl}\mathbf{F} &= N_x - M_y \\
&= \omega - (-\omega) \\
&= 2\omega.
\end{aligned}$$

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