Changing Variables in Multiple Integrals

4. Changing coordinates in triple integrals

Here the coordinate change will involve three functions

$$u = u(x, y, z), \quad v = v(x, y, z) \quad w = w(x, y, z)$$

but the general principles remain the same. The new coordinates u, v, and w give a three-dimensional grid, made up of the three families of contour surfaces of u, v, and w. Limits are put in by the kind of reasoning we used for double integrals. What we still need is the formula for the new volume element dV.

To get the volume of the little six-sided region ΔV of space bounded by three pairs of these contour surfaces, we note that nearby contour surfaces are approximately parallel, so that ΔV is approximately a parallelepiped, whose volume is (up to sign) the 3×3 determinant whose rows are the vectors forming the three edges of ΔV meeting at a corner. These vectors are calculated as in section 2; after passing to the limit we get

$$dV = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw \;, \tag{24}$$

where the key factor is the **Jacobian**

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \;. \tag{25}$$

As an example, you can verify that this gives the correct volume element for the change from rectangular to spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi; \tag{26}$$

while this is a good exercise, it will make you realize why most people prefer to derive the volume element in spherical coordinates by geometric reasoning.