Problems: Flux Through General Surfaces

1. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{k}$ and let S be the graph of $z = x^2 + y^2$ above the unit square in the xy-plane. Find the *upward flux* of $\mathbf{F}$ through S.

Answer: We can save time by noting that $\mathbf{F}$ is a tangential vector field and the vectors in $\mathbf{F}$ are parallel to S.

Otherwise, for a surface $z = f(x, y)$ we know that (for the upward normal)

$$\mathbf{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy.$$

In this case, $\mathbf{n}\,dS = \langle -2x, -2y, 1\rangle\,dx\,dy$.

Then $\mathbf{F}\cdot\mathbf{n}\,dS = (2xy - 2xy)\,dx\,dy = 0\,dx\,dy$.

Hence, Flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 0$.

2. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{k}$ and let S be the graph of $z = x^2 + y$ above the square with vertices at $(0, 0, 0)$, $(2, 0, 0)$, $(2, 2, 0)$ and $(0, 2, 0)$. Find the upward flux of $\mathbf{F}$ through S.

Answer:

Figure 1: The surface $z = x^2 + y$.

Step 1. Find $\mathbf{n}\,dS$: Here $\mathbf{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy = \langle -2x, -1, 1\rangle\,dx\,dy$.

Step 2. $\mathbf{F}\cdot\mathbf{n}\,dS = \langle -y, x, 0\rangle\cdot\langle -2x, -1, 1\rangle\,dx\,dy = (2xy - x)\,dx\,dy$.

Step 3. Flux $= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R (2xy - x)\,dx\,dy$, where R is the region in the xy-plane below S, i.e. the region 'holding' the parameters x and y.

Step 4. Compute the integral:

Limits: inner x: from 0 to 2, outer y: from 0 to 2.

$\Rightarrow$ flux $= \displaystyle\int_0^2\int_0^2 2xy - x\,dx\,dy.$

Inner: $2(2y - 1)$.

Outer: $2\left(y^2 - y\right)\Big|_0^2 = 4 =$ upward flux.

Note that this implies that the *downward flux* is -4; upward and downward flux are about the choice of $\mathbf{n}$, not $\mathbf{F}$.