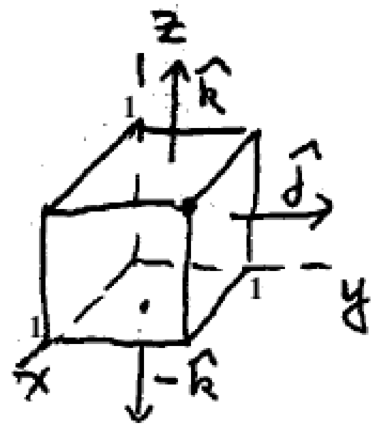
Problems: Del Notation; Flux

1. Verify the divergence theorem if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of the unit cube with opposite vertices $(0, 0, 0)$ and $(1, 1, 1)$.

Answer: To confirm that $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_D \text{div}\mathbf{F}\, dV$ we calculate each integral separately. The surface integral is calculated in six parts – one for each face of the cube.



Flux through top: $\mathbf{n} = \mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\, dS = z\, dx\, dy = dx\, dy$

$$\Rightarrow \iint_{\text{top}} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 dx\, dy = 1.$$

bottom: $\mathbf{n} = -\mathbf{k} \Rightarrow \mathbf{F}\cdot\mathbf{n}\, dS = -z\, dx\, dy = 0\, dx\, dy \Rightarrow \iint_{\text{bottom}} \mathbf{F}\cdot\mathbf{n}\, dS = 0.$

right: $\mathbf{n} = \mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n}\, dS = y\, dx\, dz = dx\, dz$

$$\Rightarrow \iint_{\text{right}} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 dx\, dz = 1.$$

left: $\mathbf{n} = -\mathbf{j} \Rightarrow \mathbf{F}\cdot\mathbf{n}\, dS = -y\, dx\, dz = 0\, dx\, dz \Rightarrow \iint_{\text{left}} \mathbf{F}\cdot\mathbf{n}\, dS = 0.$

front: $\mathbf{n} = \mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n}\, dS = x\, dy\, dz = dy\, dz$

$$\Rightarrow \iint_{\text{front}} \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1\int_0^1 dy\, dz = 1.$$

back: $\mathbf{n} = -\mathbf{i} \Rightarrow \mathbf{F}\cdot\mathbf{n}\, dS = -x\, dy\, dz = 0\, dy\, dz \Rightarrow \iint_{\text{back}} \mathbf{F}\cdot\mathbf{n}\, dS = 0.$

The total flux through the surface of the cube is 3. (We could have used geometric reasoning to see that the flux through the back, left and bottom sides is 0; the vectors of F are parallel to the surface along those sides.)

To calculate the divergence we start by noting $\text{div}\mathbf{F} = 1 + 1 + 1 = 3$. Then

$$\iiint_D \text{div}\mathbf{F}\, dV = \iiint_D 3\, dV = 3\cdot(\text{Volume}) = 3.$$

We have verified that $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_D \text{div}\mathbf{F}\, dV$ in this example.

2. Prove that $\frac{1}{2}\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{F}) = \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F}) + (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}$, where $\langle P, Q, R\rangle\cdot\boldsymbol{\nabla}$ is the differential operator $P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$.

Answer: We expand the left hand side, then the right hand side, then note that the expansions are equal. As usual, we assume $\mathbf{F} = \langle P, Q, R\rangle$.

LHS:

$$\begin{aligned}
\frac{1}{2}\boldsymbol{\nabla}\left(\mathbf{F}\cdot\mathbf{F}\right) &= \frac{1}{2}\boldsymbol{\nabla}(P^2+Q^2+R^2) \\
&= \left(P\frac{\partial P}{\partial x}+Q\frac{\partial Q}{\partial x}+R\frac{\partial R}{\partial x}\right)\mathbf{i}+\left(P\frac{\partial P}{\partial y}+Q\frac{\partial Q}{\partial y}+R\frac{\partial R}{\partial y}\right)\mathbf{j} \\
&\quad +\left(P\frac{\partial P}{\partial z}+Q\frac{\partial Q}{\partial z}+R\frac{\partial R}{\partial z}\right)\mathbf{k}.
\end{aligned}$$

RHS (in two parts):

$$\begin{aligned}
\mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F}) &= \mathbf{F}\times\left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathbf{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathbf{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathbf{k}\right] \\
&= \left(Q\frac{\partial Q}{\partial x}-Q\frac{\partial P}{\partial y}-R\frac{\partial P}{\partial z}+R\frac{\partial R}{\partial x}\right)\mathbf{i} \\
&\quad +\left(R\frac{\partial R}{\partial y}-R\frac{\partial Q}{\partial z}-P\frac{\partial Q}{\partial x}+P\frac{\partial P}{\partial y}\right)\mathbf{j} \\
&\quad +\left(P\frac{\partial P}{\partial z}-P\frac{\partial R}{\partial x}-Q\frac{\partial R}{\partial y}+Q\frac{\partial Q}{\partial z}\right)\mathbf{k} \\
&= \left(Q\frac{\partial Q}{\partial x}+R\frac{\partial R}{\partial x}\right)\mathbf{i}+\left(R\frac{\partial R}{\partial y}+P\frac{\partial P}{\partial y}\right)\mathbf{j}+\left(P\frac{\partial P}{\partial z}+Q\frac{\partial Q}{\partial z}\right)\mathbf{k} \\
&\quad -\left(Q\frac{\partial P}{\partial y}+R\frac{\partial P}{\partial z}\right)\mathbf{i}-\left(R\frac{\partial Q}{\partial z}-P\frac{\partial Q}{\partial x}\right)\mathbf{j}-\left(P\frac{\partial R}{\partial x}+Q\frac{\partial R}{\partial y}\right)\mathbf{k}.
\end{aligned}$$

$$\begin{aligned}
(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F} &= \left(P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}\right)\mathbf{F} \\
&= \left(P\frac{\partial P}{\partial x}+Q\frac{\partial P}{\partial y}+R\frac{\partial P}{\partial z}\right)\mathbf{i}+\left(P\frac{\partial Q}{\partial x}+Q\frac{\partial R}{\partial y}+R\frac{\partial Q}{\partial z}\right)\mathbf{j} \\
&\quad +\left(P\frac{\partial R}{\partial x}+Q\frac{\partial Q}{\partial y}+R\frac{\partial R}{\partial z}\right)\mathbf{k}.
\end{aligned}$$

Note that the negative terms in $\mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{F})$ are cancelled by positive terms in $(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}$, leading to the desired result.