Problems: Gradient Fields and Potential Functions

Is the differential $y^2z\,dx + 2xyz\,dy + xy^2\,dz$ exact? If so, find a potential function. If not, explain why not.

Answer: The differential $f = M\,dx + N\,dy + P\,dz$ is exact if M, N, and P are continuously differentiable in all of 3-space and $P_y = N_z$, $M_z = P_x$ and $N_x = M_y$. Here $M = y^2z$, $N = 2xyz$ and $P = xy^2$ are continuously differentiable in all of 3-space. We compute:

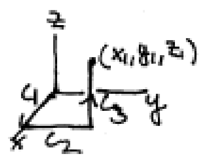
$$P_y = 2xy, \qquad N_z = 2xy,$$

$$M_z = y^2, \qquad P_x = y^2,$$

$$N_x = 2yz, \qquad M_y = 2yz.$$

This confirms that the differential is exact and so equal to df for some function $f(x, y, z)$. (Note that this method is equivalent to showing that $\operatorname{curl}\langle M, N, P\rangle = \mathbf{0}$.)

To find a potential function we could "guess and check" or integrate along a path like the one shown below. (We carefully chose the path to make the integrals as simple as possible.)



$$\int_0^{x_1} 0\,dx + \int_0^{y_1} 0\,dy + \int_0^{z_1} x_1y_1^2\,dz = x_1y_1^2z_1.$$

We conclude that the potential functions for this differential are of the form $f = xy^2z + C$.