JOEL LEWIS: Hi. Welcome back to recitation. In lecture, you've been learning about Stokes' Theorem, and I have a nice exercise on Stokes' Theorem for you here.

So I'm going to let F be this field that I've written just above me. So it's 2x*z minus 2y comma 2y*z plus 2x comma x square plus y square plus z square. And I've got C.

So C is this complicated-looking curve here. So it sort of dips up and down and back around. But the thing I'm really going to tell you about it is that it all lies on this cylinder of radius b. So C is this curve in the cylinder of radius b that wraps around it once, but behaves kind of oddly while it's wrapping around.

So what I'd like you to do is I'd like you to use Stokes' Theorem to compute the integral around this curve of F dot dr. Now, my hint to you is that for Stokes' Theorem, you can use-- just like you have for Green's Theorem and for Divergence Theorem that we've talked about before, you have these extended versions that let you consider more than one boundary piece. So the same thing works for Stokes' Theorem.

So Stokes' Theorem works perfectly well when you have a piece of a surface with more than one boundary curve, provided you orient everything correctly. So you might think about how you can use Stokes' Theorem to replace this complicated curve with a surface integral and an easier to understand curve. And if you can do that, then computing the other two gives you the third one.

All right. So that's my hint to you for computing this integral. So why don't you pause the video, have a go at that, come back, and we can work on it together.

Hopefully, you had some luck working on this problem. Let's talk about it. So before I left, I gave you this hint that maybe the thing to do here isn't to try and parametrize this curve directly and compute the line integral directly since it's a complicated-looking curve, and also since I haven't really given you enough information to do that, and instead to think about applying Stokes' Theorem.

So to think about applying Stokes' Theorem, what we'd like is a nice surface, with this curve as part of its boundary. Well, what is such a surface? Well, this curve lies all on the cylinder of radius b. So a natural choice for a surface is to use some piece of this cylinder.

So maybe we could use the piece of this cylinder with this as its upper boundary. So then what might be a natural lower boundary to choose? Well, we just want to choose something nice and simple. Right? So, what's nice and simple? Well, maybe we can choose this bottom circle that's in the plane y equals x. All right. So I'm going to call that circle C_1.

So that's the circle of radius b in the xy-plane. Sorry, not the plane y equals x. The xy-plane. The plane z equals 0.

So we've got the top curve C and we've got this bottom curve C_1. Now, the way I've oriented them, I've oriented them both so that they're going counterclockwise as you look down from the z-axis. So in that case, what does Stokes' Theorem say?

Well, Stokes' Theorem says that the integral over the piece of the surface between them-- let's call it S-- of curl F dot n with respect to surface area is equal to-- OK. So let's say we can give it the outward pointing normal, say. In which case, C_1 will be positively oriented and C will be negatively oriented. So this is equal to the line integral over C_1 of F dot dr minus the line integral over C of F dot dr.

And so what's nice about this formula is that it replaces computing the integral that we want. Instead of computing that, we can try and compute this other line integral and this surface integral. And if these are easier to compute, then computing the two of them gives us what the value of this is just by subtracting, or by adding and subtracting, or whatever. By arithmetic, right? So if these integrals are easy to compute, then that makes this one easy without actually having to parametrize and compute it.

So let's take a look at what these integrals are. Let's do the surface integral first since it's on the left. So in order to compute the surface integral, we're going to need to compute the curl of F.

So OK. So F is this kind of messy-looking thing here. So curl of F, well, what have we got? So it's going to be big thing times i hat. So it's going to be i hat times this determinant, right? So let me write the determinant.

So on top we've got i hat, j hat, k hat, then we have the partial x, partial y, partial z, and then we have the components. So these are 2x*z minus 2y, and 2y*z plus 2x, and x square plus y square plus z square. All right.

So that's i hat, j hat, and k hat. And then we've got partial over partial x, partial over partial y, and partial over partial z. So this is what the curl is, and so now we have to expand this out.

So for i, it's going to be partial y of x squared plus y squared plus z squared-- so that's 2y-- minus partial z of 2y*z plus 2x, so that's minus 2y, i hat. Plus-- for j hat, it's going to be partial z of 2x*z minus 2y, so that's 2x-- minus partial x of x squared plus y squared plus z squared, so that's minus 2x, j hat. Plus-- for k hat, we want partial x of 2y*z plus 2x, so that's 2-- minus partial y of 2x*z minus 2y, so that's minus minus 2, so that's plus 2, k hat.

Oh. All right. OK. So the i-component is 0 and the j-component is 0. So this is a nice, simple one. So the curl here, the k-component is just 4. So this is equal to 4k. OK. So that's what the curl of F is.

Now what do we need to compute? We need to compute curl of F dot the normal vector with respect to surface area. Now let's look at what our surface is.

Our surface is right here, and it's this vertical cylinder. Well, what is the normal vector of a vertical cylinder? Well, it's pointing straight away from the axis. Right? It's perpendicular to the surface of the cylinder, so it's parallel to the xy-plane. It rotates as you go around the cylinder, but it's always in the xy-plane.

So what does that mean? Well, that means in particular, it's perpendicular to things in the z-direction. Right? So if we look, we see our curl here is just straight upward in the z-direction. And our normal vector has no z-component. It's only in the xy-plane.

So this k hat is orthogonal to n, OK? So the curl and n are orthogonal. So their dot product is 0. So this surface integral is a surface integral of 0. So it just gives you 0.

OK. So great. So that's really nice. That simplifies our life very much. Now, our line integral that we want. We just have it in terms of this one other line integral. Right? So the surface integral is 0. And let me see. Where should I put this?

OK, the curl is 4k, so the surface integral curl F dot n dS is also equal to 0. So having made that simplification, now we just need this other integral. We need this line integral over C_1. And that'll give us what we need. So let's have a go at that.

So C_1 is the circle of radius b centered at the origin in the xy-plane. OK. I'm not going to write that down. I'm just going to say it. The circle of radius b centered at the origin in the xy-plane. So OK. So it's not that hard to parametrize.

So it's parametrized by x equals b cosine theta, y equals b sine theta. We should check. We should double-check that we're doing the right direction of parametrization. Let's go have a look. Let's see.

Yes. OK. So we parametrized this circle going counterclockwise in the xy-plane. So good. So this parametrization is going the right direction. Otherwise, we'd have to change the sign of theta or something.

So it's x is b cosine theta, y is b sine theta. And we're going once around the circle, so we want 0 less than or equal to theta less than or equal to 2*pi. And so what do we have? So now, we want to compute the integral over the circle of F dot dr. So let's see what F looks like in this situation. So let's go back and look at the expression for F over here.

So in this plane, we have z is equal to 0. So F is minus 2y, plus 2x, x squared plus y squared. OK? OK, so let's come back then.

So F is what I just said, so this is equal to the integral over C of minus 2y dx, plus 2x dy-- plus x squared plus y squared dz, but we're in the plane z equals 0, so dz is always 0 in that plane-- so we don't have a third term there. Great. So this is our integral, and now we can substitute from our parametrization here.

So this is equal to the integral from 0 to 2*pi. So minus 2y dx. So that's minus 2b sine theta, times-- dx is minus b sine theta d theta-- plus 2x-- so that's 2b cosine theta-- times dy, which is b cosine theta d theta. Whew. This is quite a long equation, isn't it? Or a long expression, I guess.

So our line integral around C of F dot dr is equal to the integral from 0 to 2*pi of minus 2b sine theta times minus b sine theta d theta. So this is 2b squared sine squared theta d theta. And this is 2b cosine squared theta d theta.

So OK. So that 2 b squared is a constant. We can just factor it out. And we're left with sine squared theta plus cosine squared theta d theta. All right. OK. That's great. I'm happy to have that. Right? Sine squared theta plus cosine squared theta, that's going to be 1. OK.

So we can rewrite this. I'm going to bring it back up here. So that's equal to the integral from 0 to 2*pi of 2 b squared d theta, which is 4*pi b squared. Great. OK, so that's our line integral around this bottom curve C.

Oh, dear. I've been writing C, but this is not our original curve C, this is our new curve C_1, like I wrote there. Sorry. So everywhere I wrote the line integral over C-- both of these places-- it was supposed to be a line integral over C_1. Sorry about that.

So we've got this line integral over C_1, and it worked out to 4*pi b squared, just using our usual parametrize-and-compute technique for computing line integrals. So OK. So now, let's see where we're at. Let's go back over here to when we wrote down what the extended Stokes' Theorem says in our case.

So Stokes' Theorem told us that the thing we were interested in-- this is the thing we're trying to compute, right? The problem asked us to compute the line integral over C of F dot dr. Well, extended Stokes' Theorem said, in order to compute this line integral, what you can do is you can compute this surface integral over S, and you can compute this line integral over this other curve C_1, and then these three things have to satisfy this relationship. That's what's Stokes' Theorem says.

And now we've computed. We've computed the surface integral, and we found it was equal to 0 by a simple geometric argument that didn't require us to actually compute a surface integral. And we computed this line integral, just now, by parametrizing and computing it. So OK.

So this was 0 and this was 4*pi b squared. So if we just add our integral in question to the other side, what we find-- I'm going to go find some empty board space to write it down-- so our integral, the integral over C of F dot dr is equal to this other line integral minus the surface integral. So it's equal to 4*pi b squared minus 0. Just rearranging that equation we were looking at a second ago from Stokes' Theorem. So it's just 4*pi b squared.

So that's the answer, and I'll end there.