18.03 Problem Set 1

I encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

Part I of each problem set will consist of problems which are either rather routine or for which solutions are available in the back of the book, or in 18.03 *Notes and Exercises*.

Part II contains more challenging and novel problems. They will be graded with care (Complain if they are not!) and contribute the bulk of the Homework grade. They will help you develop an understanding of the material.

Problems in both parts are keyed closely to the lectures, and numbered to match them. Try the problems as soon as you can after the indicated lecture. Most problem sets correspond to four lectures, through the Monday or Wednesday before the set is due. Each problem set is graded out of 100 points. Each day counts 24 points. The grader has 4 points to give or withhold based on neatness and clarity of the answers. (Problem sets 6 and 7 relate to only three lectures, and each day in them counts 32 points.)

I. First-order differential equations

R1	T 2 Feb	Natural growth models and separable equations: EP 1.1, 1.4.
L1	W 3 Feb	Direction fields, existence and uniqueness of solutions: EP 1.2, 1.3; Notes G.1; SN 1.
R2	Th 4 Feb	Isoclines, separatrix, extrema of solutions: Notes G.1, D.
L2	F 5 Feb	Numerical methods: EP 6.1, 6.2; Notes G.2.
L3	M 8 Feb	Linear equations: models: EP 1.5; SN 2.
R3	T 9 Feb	Mixing problems, half-life.
L4	W 10 Feb	Solution of linear equations, integrating factors: EP 1.5; SN 3.

Part I.

0. (T 2 Feb) [Natural growth, separable equations] Notes 1A-5c; EP 1.1: 32, 33, 35; EP 1.4: 39, 66; EP 1.5: 1, 9, 20.

1. (W 3 Feb) [Direction fields, isoclines] Notes 1C-1abe.

2. (F 5 Feb) [Euler's method] Notes 1C-4.

3. (M 8 Feb) [Linear models] EP 1.5: 33, 45. (In both, be sure to write out the differential equation.)

Part II.

0. (T 2 Feb) [Natural growth, separable equations] In recitation a population model was studied in which the natural growth rate of the population of oryx was a constant $k > 0$, so that for small time intervals Δt the population change $x(t+\Delta t) - x(t)$ is well approximated by $kx(t)\Delta t$. (You also studied the effect of hunting them, but in this problem we will leave that aside.) Measure time in years and the population in kilo-oryx (ko).

A mysterious virus infects the oryxes of the Tana River area in Kenya, which causes the growth rate to decrease as time goes on according to the formula $k(t) = k_0/(a+t)^2$ for $t \geq 0$, where a and k_0 are certain positive constants.

(a) What are the units of the constant a in "$a + t$," and of the constant k_0?

(b) Write down the differential equation modeling this situation.

(c) Write down the general solution to your differential equation. Don't restrict yourself to the values of t and of x that are relevant to the oryx problem; take care of all values of these variables. Points to be careful about: use absolute values in $\displaystyle\int \frac{dx}{x} = \ln|x| + c$ correctly, and don't forget about any "lost" solutions.

(d) Now suppose that at $t = 0$ there is a positive population x_0 of oryx. Does the progressive decline in growth rate cause the population stabilize for large time, or does it grow without bound? If it does stabilize, what is the limiting population as $t \to \infty$?

1. (W 3 Feb) [Direction fields, isoclines] In this problem you will study solutions of the differential equation

$$\frac{dy}{dx} = y^2 - x\,.$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) Draw a large pair of axes and mark off units from -4 to $+4$ on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -1$, $m = 0$, $m = 1$, and $m = 2$. On this same graph, sketch, as best you can, a couple of solutions, using just the information given by these four isoclines.

Having done this, you will continue to investigate this equation using one of the Mathlets. So invoke `http://math.mit.edu/mathlets/mathlets` in a web browser and select `Isoclines` from the menu. (To run the applet from this window, click the little black box with a white triangle inside.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with a brief description of the applet's functionalities.

Select from the pull-down menu our differential equation $y' = y^2 - x$. Move the m slider to $m = -2$ and release it; the $m = -2$ isocline is drawn. Do the same for $m = 0$, $m = 1$, and $m = 2$. Compare with your sketches. Then depress the mousekey over the graphing window and drag it around; you see a variety of solutions. How do they compare with what you drew earlier?

(b) A separatrix is a curve such that above it solutions behave (as x increases) in one way, while below it solutions behave (as x increases) in quite a different way. There is a separatrix for this equation such that solutions above it grow without bound (as x increases) while solutions below it eventually decrease (as x increases). Use the applet to find its graph, and submit a sketch of your result.

(c) Suppose $y(x)$ is a solution to this differential equation whose graph is tangent to the $m = -1$ isocline: it touches the $m = -1$ isocline at a point (a, b), and the two curves have the same slope at that point. Find this point on the applet, and then calculuate the values of a and b.

(d) Now suppose that $y(x)$ is a solution to the equation for which $y(a) < b$, where (a, b) is the point you found in **(c)**. What happens to it as $x \to \infty$? More specifically, give an explicit function $f(x)$ whose graph is asymptotic to the graph of the function $y(x)$. For large x, is $y(x) > f(x)$, $y(x) < f(x)$, or does the answer depend on the value of $y(a)$?
The following observations will be useful in justifying your claims. Please explain as clearly as you can why each is true.
(i) The graph of $y(x)$ can't cross the $m = -1$ isocline at a point (x, y) with $x > a$.
(ii) If the graph of $y(x)$ is below the $m = 1$ isocline for $x = a$, it will eventually (as x grows) cross the $m = 1$ isocline.
(iii) The graph of $y(x)$ must eventually cross the nullcline.

(e) Suppose a solution $y(x)$ has a critical point at (c, d)—that is, $y'(c) = 0$ and $y(c) = d$. What can you say about the relationship between c and d? The applet can be very helpful here, but verify your answer.

(f) It appears from the applet that all critical points are local maxima. Is that true?

2. (F 5 Feb) [Euler's method] **(a)** Write y for the solution to $y' = 2x$ with $y(0) = 0$. What is $y(1)$? What is the Euler approximation for $y(1)$, using 2 equal steps? 3 equal steps? What about n steps, where n can now be any natural number? (It will be useful to know that $1 + 2 + \cdots + (n - 1) = n(n - 1)/2$.) As $n \to \infty$, these approximations should converge to $y(1)$. Do they?

(b) In the text and in class it was claimed that for small h, Euler's method for stepsize h has an error which is at most proportional to h. The n-step approximation for $y(1)$ has $h = 1/n$. What is the exact value of the difference between $y(1)$ and the n-step Euler approximation? Does this conform to the prediction?

3. (M 8 Feb) [Linear models] Scrooge McDuck wants to set up a trust fund for his nephew Don. He has fool-proof investments which make a constant interest rate of I, measured in units of $(\text{years})^{-1}$ (so $I = 0.05$ means 5% per year), and he proposes to dole out the money to his profligate nephew at a constant rate q dollars per year.

(a) Model this process by a differential equation. (Use the symbols I and q, rather than specific values for them.) Explain your steps.

(b) Then find the general solution to this differential equation.

(c) Now take $I = 0.05$. If Uncle Scrooge wanted to fund the trust so as to provide his nephew with \$1000 per *month* in perpetuity, while maintaining a constant balance in the fund, how much should he invest?

(d) But in fact Uncle Scrooge wants to teach his nephew the virtues of self-reliance, and so plans on having the trust fund run entirely out of money in exactly twenty years. If he wants to give his nephew \$1000 per month, how much should he fund the trust with at the outset? Give the answer to the nearest penny (as Scrooge would insist on).

18.03 Differential Equations
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