18.03 Problem Set 2

I encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

I. First-order differential equations

L4	W 13 Feb	Solution of linear equations; integrating factors: EP 1.5, SN 3.
R4	Th 14 Feb	Ditto
L5	F 15 Feb	Complex numbers, roots of unity: SN 5–6; Notes C.1–3.
L6	T 19 Feb	Complex exponentials; sinusoidal functions: Notes C.4; SN 4; Notes IR.6.
L7	W 20 Feb	Linear system response to exponential and sinusoidal input; gain, phase lag: SN 4, Notes IR.6.
R5	Th 21 Feb	Sinusoids; complex numbers and exponentials.
L8	F 22 Feb	Autonomous equations; the phase line, stability: EP 1.7, 7.1.

Part I.

4. (W 13 Feb) [Solutions to linear equations] EP 1.5: 1, 2, 5. Remember to check whether the equation is separable first.

5. (F 15 Feb) [Complex numbers, roots of unity] Notes 2E-1, 2, 7.

6. (T 19 Feb) [Complex exponentials, Sinusoids] **(a)** Notes 2E-9, 10.
(b) Write each of the following functions $f(t)$ in the form $A\cos(\omega t - \phi)$. In each case, begin by drawing a right triangle as in the Supplementary Note SN4 (on the course website) and as in Lecture 6.
(i) $\cos(2t) + \sin(2t)$. (ii) $\cos(\pi t) - \sqrt{3}\sin(\pi t)$. (iii) $\cos(t - \frac{\pi}{8}) + \sin(t - \frac{\pi}{8})$.

7. (W 20 Feb) [First order harmonic response] **(a)** 2E-15.
(b) (i) Find a solution of $\dot{x} + 2x = e^{3t}$ of the form Be^{3t}. Then write down the general solution. (ii) Now do the same for the complex-valued differential equation $\dot{x} + 2x = e^{3it}$.

Part II.

4. (W 13 Feb) [Solutions to linear equations] Almost all the radon in the world today was created within the past week or so by a chain of radioactive decays beginning mainly from uranium, which has been part of the earth since it was formed. This cascade of decaying elements is quite common, and in this problem we study a "toy model" in which the numbers work out decently. This is about Tatooine, a small world endowed with unusual elements.

A certain isotope of Startium, symbol St, decays with a half-life t_S. Strangely enough, it decays with equal probability into a certain isotope of either Midium, Mi, or into the little known stable element Endium. Midium is also radioactive, and decays with half-life t_M into Endium. All the St was in the star-stuff that condensed into Tatooine, and all the Mi and En arise from the decay route described. Also, $t_M \neq t_S$.

Use the notation $x(t)$, $y(t)$, and $z(t)$, for the amount of St, Mi, and En on Tatooine, in units so that $x(0) = 1$.

(a) Make rough sketches of graphs of x, y, z, as functions of t. What are the limiting values as $t \to \infty$?

(b) Write down the differential equations controlling x, y, and z. Be sure to express the constants that occur in these equations correctly in terms of the relevant decay constants. Use the notation σ (Greek letter sigma) for the decay constant for St and μ (Greek letter mu) for the decay constant for Mi. Your first step is to relate σ to t_S and μ to t_M. A check on your answers: the sum $x+y+z$ is constant, and so we should have $\dot{x}+\dot{y}+\dot{z} = 0$.

(c) Solve these equations, successively, for x, y, and z.

(d) At what time does the quantity of Midium peak? (This will depend upon σ and μ.)

(e) Suppose that instead of $x(0) = 1$, we had $x(0) = 2$. What change will this make to $x(t)$, $y(t)$, and $z(t)$?

(f) Unrelated question: Suppose that $x(t) = e^t$ is a solution to the differential equation $t\dot{x} + 2x = q(t)$. What is $q(t)$? What is the general solution?

5. (F 15 Feb) [Complex numbers, roots of unity] **(a)** Make a table with three columns. Each row will contain three representations of a complex number z: the "rectangular" expression $z = a + bi$ (with a and b real); the "polar" expression $|z|$, $\mathrm{Arg}(z)$; and a little picture of the complex plane with the complex number marked on it. There are five rows, containing, in one column or another, the following complex numbers:
(i) $1 - i$
(ii) z such that $|z| = 2$ and $\mathrm{Arg}(z) = \pi/6$
(iii) A square root of i with negative imaginary part
(iv) A sixth root of 1 with argument θ such that $0 < \theta < \pi/2$
(v) $((1+i)/\sqrt{2})^{-13}$.

(b) Find the complex roots of the following equations: $z^4 + 4 = 0$; $z^2 + 2z + 2 = 0$.

6. (T 19 Feb) [Complex exponentials, Sinusoids] **(a)** Add a fourth column to table you made in **5 (a)** by giving the exponential representation $z = Ae^{i\theta}$ (with A and θ real).

(b) Find all complex numbers $z = a + bi$ such that $e^z = -2$.

(c) Find an expression for $\cos(4t)$ in terms of sums of powers of $\sin t$ and $\cos t$ by using $(e^{it})^4 = e^{4it}$ and Euler's formula.

(d) The Mathlet `Complex Exponential` will probably be useful in understanding the rest of this problem. Open it and explore its functionalities. The `Help` button lists most of them. Notice that in the left window, the real part a ranges between -1 and 1, while the imaginary part b ranges from -8 to 8. You use the left-hand window to pick out a complex number $a+bi$. When you do, a portion of the line through it and zero is drawn. This line is parametrized by $(a+bi)t$. At the same time, the curve parametrized by the complex-valued function $e^{(a+bi)t}$ is drawn on the right window.

For each of the following functions $f(t)$, make a sketch of the graph (in a convenient range), find a value of $w = a+bi$ such that $\mathrm{Re}(e^{wt}) = f(t)$, and sketch the trajectory of the complex valued function e^{wt} (i.e. its set of values).

(i) $f(t) = \cos(2\pi t)$

(ii) $f(t) = e^{-t}$

(iii) $f(t) = e^{-t}\cos(2\pi t)$.

(iv) The constant function with value 1.

7. (W 20 Feb) [Sinusoidal input and output] **(a)** Express $\mathrm{Re}\left(\dfrac{e^{3it}}{\sqrt{3}+i}\right)$ in the form $a\cos(3t) + b\sin(3t)$. Then rewrite this in the form $A\cos(3t-\phi)$. Now find this same answer by a different route: By finding its modulus and argument, write $2+2i$ in the form $Ae^{i\phi}$. Substitute this into $e^{2it}/(2+2i)$, and then use properties of the exponential function to find B and ϕ such that $\dfrac{e^{3it}}{\sqrt{3}+i} = Be^{i(3t-\phi)}$. Finally, take the real part of this new expression.

(b) Find a solution of the differential equation $\dot{z} + 3z = e^{2it}$ of the form we^{2it}, where w is some complex number. What is the general solution?

(c) Finally, find a solution of $\dot{x} + 3x = \cos(2t)$ by relating this ODE to the one in **(b)**. What is the general solution?

Answers

I. 6 (b): (i) $\sqrt{2}\cos(2t - \frac{\pi}{4})$.
(ii) $2\cos(\pi t + \frac{\pi}{3})$.
(iii) $\sqrt{2}\cos(t - \frac{3t}{8})$.

I. 7 (b) (i) $\dfrac{e^{3t}}{5} + ce^{-2t}$.

(ii) $\dfrac{e^{3it}}{2+3i} + ce^{-2t}$.