

18.034 EXAM 2
MARCH 17, 2004

Name: _____

Problem 1: _____ /30

Problem 2: _____ /20

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Problem 4: _____ /25

Total: _____ /100

Instructions: *Please write your name at the top of every page of the exam.* The exam is closed book, closed notes, and calculators are not allowed. You will have approximately 50 minutes for this exam. The point value of each problem is written next to the problem – use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown. You may use either pencil or ink. If you have a question, need extra paper, need to use the restroom, etc., raise your hand.

Name: _____

Problem 1: _____ /30

Problem 1(30 points) A driven, damped harmonic oscillator satisfies the following ODE,

$$y'' + 2by' + \omega^2 y = F \cos(\omega t),$$

where b , ω and F are positive real numbers.

(a)(10 points) Using the method of undetermined coefficients, find a solution of the form $y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$.

(b)(10 points) Let R be a positive real number, $R \leq F/(2b\omega)$ (this guarantees that $y_d(t) = R$ for some $t > 0$). There is a multi-valued function $T = T(\omega)$ for the set of positive numbers where $y(T) = R$. This can be made into a single-valued function T_n by specifying that T_n is the n^{th} smallest positive number such that $y(T) = R$. So T_1 is the smallest positive number such that $y(T_1) = R$, T_2 is the smallest positive number greater than T_1 such that $y(T_2) = R$, etc.

For at least one choice of $n > 0$, find a formula for $T_n(\omega)$.

(c)(10 points) Suppose that b , F and R are fixed, but ω is allowed to vary,

$$0 < \omega \leq \frac{F}{2bR}.$$

Write $T = T_n$ for some positive integer n . Let ω be a critical point of $T(\omega)$. Prove there is an equation,

$$\omega T(\omega) = \alpha,$$

where α is a real number independent of b , F and R . Moreover, find an equation that α satisfies. (**Remark** You will see there are many choices for α . You are not responsible for “matching” choices of α to choices of n ; just write down an equation that α satisfies).

Extra credit(5 points) Prove the solutions of your equation for α give local minima of $T(\omega)$.

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Problem 2: _____ /20

Problem 2(20 points) In each case below, $y_1(t)$, $y_2(t)$ is a pair of solutions of a real, constant coefficient, linear homogeneous ODE in normal form. Determine the least degree of this ODE, and write down the ODE of this degree that the pair satisfies. (**Hint:** In each case, write down each nonzero solution as the real or imaginary part of $e^{\lambda t}g(t)$ where $g(t)$ is a polynomial. What does the degree of $g(t)$, and the vanishing/nonvanishing of the imaginary part of λ tell you about the characteristic equation of the ODE? Remember, the ODE is a *real* ODE.)

(a)(5 points) $y_1(t) = 0$, $y_2(t) = e^t$.

(b)(5 points) $y_1(t) = e^{-t}$, $y_2(t) = e^{-2t}$.

(c)(5 points) $y_1(t) = t$, $y_2(t) = e^t$.

(d)(5 points) $y_1(t) = \sin(2t)$, $y_2(t) = \cos(3t)$.

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Problem 3: _____ /25

Problem 3(25 points) For a certain linear ODE in normal form Ly a basic solution set of $Ly = 0$ is given by,

$$y_1(t) = e^{2t}, \quad y_2(t) = 2t^2 + 2t + 1.$$

(a)(10 points) Compute the Wronskian of this basic solution pair. Is your answer consistent with Abel's theorem?

(b)(15 points) Using the method of variation of parameters, find a particular solution of the inhomogeneous ODE,

$$Ly = t^2 e^{2t}.$$

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Problem 4: _____ /25

Problem 4(25 points) One solution of the linear ODE,

$$y'' + a(t)y' + b(t)y =$$

$$y'' + \left(-2 - \frac{2}{t}\right)y' + \frac{4}{t}y = 0,$$

is the equation $y_1(t) = e^{2t}$.

(a)(10 points) Let $y_2(t)$ be a second solution. Define $W(t)$ to be the Wronskian of $y_1(t), y_2(t)$. This satisfies the differential equation $W' = -a(t)W$. Find a solution $W(t)$ of this differential equation.

(b)(15 points) For your solution $W(t)$, solve the first-order linear ODE,

$$y_1(t)v' - y_1'(t)v = W(t).$$

Plug in $y_2(t) = v$ to find a basic solution pair of the ODE, $Ly = 0$.