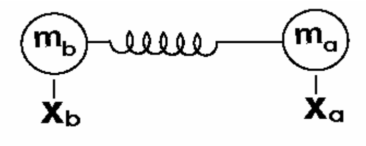
18.034, Honors Differential Equations
Prof. Jason Starr
Lecture 26
4/9/04

1. Spent about ½ lecture working through the linear system of a pair of masses connected by a spring (of equilibrium displacement L)



$$m_a x_a'' = -k(x_a - X_b - L)$$
$$m_a x_b'' = -k(x_a - X_b - L)$$

Introduce $v_a = x_a'$, $V_b = X_b'$, $X=\begin{pmatrix} x_a \\ v_a \\ x_b \\ v_b \end{pmatrix}$, $\quad \underset{=}{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{m_a} & 0 & \frac{k}{m_a} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_b} & 0 & -\frac{k}{m_b} & 0 \end{bmatrix}$, $\quad \underset{=}{F} = \begin{bmatrix} 0 \\ \frac{kL}{m_a} \\ 0 \\ -\frac{kL}{m_b} \end{bmatrix}$.

Then $x' = Ax + F$. Physics (intuition suggests introducing $M = \frac{m_a m_b}{m_a + m_b}$,

$y_1 = \frac{M}{m_b} x_a + \frac{M}{m_a} x_b$, $V_1 = y_1' = \frac{M}{m_b} x_a + \frac{M}{m_a} x_b$, $y_2 = x_a - x_b$, $V_1' = y_1' = V_a - V_b$. Then

$$y' = By + G \text{ where } B = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{k}{m} & 0 \end{array}\right], \qquad G = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ \frac{kL}{M} \end{array}\right].$$

Now (y_1, V_1) is decoupled from (y_2, V_2).

2. We paused at this point and solved the system by usual 2nd order method. But then we continued to 'diagonalize' (y_2, V_2).

Implicit C.O.V. : $\quad \begin{aligned} y_2 &= z_1 + z_2 \\ V_2 &= i\omega z_1 + i\omega z_2 \end{aligned} \quad \longrightarrow \quad z' = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} z + \begin{bmatrix} \frac{-i\omega L}{2M} \\ \frac{+i\omega L}{2M} \end{bmatrix}$, where $\omega = \sqrt{\frac{k}{M}}$

This is now completely diagonalized:

$$\begin{bmatrix} y_1 \\ v_1 \\ z_1 \\ z_2 \end{bmatrix} = y_{1,0} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_{1,0} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + z_{1,0} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{i\omega t} + z_{2,0} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-i\omega t} + \begin{bmatrix} 0 \\ 0 \\ L/2m \\ -L/2m \end{bmatrix}.$$

Back-substituting:

$$\begin{bmatrix} x_a \\ v_a \\ x_b \\ v_b \end{bmatrix} = y_{1,0}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v_{1,0}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + z_{1,0}\begin{bmatrix} M/m_a \\ i\omega\, M/m_a \\ -M/m_b \\ -i\omega\, M/m_b \end{bmatrix} e^{i\omega t} + z_{2,0}\begin{bmatrix} M/m_a \\ i\omega\, M/m_a \\ -M/m_b \\ i\omega\, M/m_b \end{bmatrix} e^{-\omega t} + \begin{bmatrix} L/m_a \\ 0 \\ -L/m_b \\ 0 \end{bmatrix}$$

eigenval = 0
eigenvector=

eigenval = 0
gen. eigenvector

eigenval = iω
eigenvect

eigenval = -iω
eigenvector

Particular sol'n.

Need to choose z_1,0 and z_2,0 to be complex conjugate to get a real solution.

3. Discussed the general eigenvector decomp. method for solving $y' = Ay$: If (λ,V) is an eigenvalue/ eigenvector pair, then $y(t) = ve^{\lambda t}$ is a sol'n.

   Defined eigenvalues + eigenvectors.

   Defined the char. poly, $\det(\lambda I - A)$.

   Saw that the eigenvalues are precisely the roots of $\det(\lambda I - A)$.

   We did not yet define/ discuss generalized eigenspaces (although I did mention the term in the solution of the 2-mass-spring problem).

1. One or two problems reducing higher order constant coeff. linear systems to 1st or order linear systems (students seemed fuzzy on this).

   Examples: (a) $y''' - 3y'' + 3y' - y = e^t$:

   $y_1 = y$
   $y_2 = y'$
   $y_3 = y''$

   $$\rightsquigarrow \quad y' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 3 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ e^t \end{bmatrix},$$

   (b) $y_1'' = y_2$
   $y_2'' = y_3$
   $y_3' = y_1 + y_2$

   $x_1 = y_1$
   $x_2 = y_1'$
   $x_3 = y_2$
   $x_4 = y_2'$
   $x_5 = y_3$

   $$\rightsquigarrow \quad x' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} x$$

2. Maybe one problem going in the other direction:

   Example: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$: $y_2 = y_1' - 2y_1$
   $y_1'' - 2y_1' = y_2' = y_1 + 2y_1' - 4y_1$

   $\rightsquigarrow$ $y_1'' - 4y_1 + 3y_1 = 0$
   $y_1 = Ae^t + Be^{3t}$
   $y_2 = -Ae^t + Be^{3t}$

3. Several diagonalizing/ eigenvalue- eigenvector problems

   e.g. (a) $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$, $\lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4)$; $\lambda = -2$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

   $\lambda = -4$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

   By exercise (18) (This week's Pset),

   $(\lambda - 1)(\lambda - 2)(\lambda - 3)$: $\lambda = 1$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $\lambda = 2$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = 3$, $\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$.

   (c) $\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$, $(\lambda + 1)^2$; $\lambda = -1$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.