18.034, Honors Differential Equations
Prof. Jason Starr
Lecture 30: <u>Notes for Kiran: I. Summary of Lecture 29</u> 4/21/04

We finished last time with Jordan normal form.

<u>Notation</u>(: i) $\mathbb{V}$ a finite dimensional $\mathbb{C}$-vector space (usually $\mathbb{V} = \mathbb{C}^n$ with standard basis $e_1, \ldots, e_n$).

(ii) $T = \mathbb{V} \to \mathbb{V}$ a linear operator

(iii) A linear system of differential equations

$$y' = Ty$$

i.e. looking for differentiable $y = \mathbb{R} \to \mathbb{V}$ s.t.

$$y'(t) = T(y(t)).$$

(iv) matrix of T w.r.t. an ordered basis $B = (v_1, \ldots, v_n)$, $A = [T]_{B,B,}$

$$Tv_j = \sum_{i=1}^{n} A_{ij} \cdot v_i .$$

(v) char. polynomial $p_T(\lambda) = \det(\lambda Id_{\mathbb{V}} - T) = \lambda^n - T_r(T)\lambda + \ldots\ldots$

(vi) factorization $p_T(\lambda) = (\lambda - \lambda_1)^{m_1} \ldots\ldots (\lambda - \lambda_t)^{m_t}$.

(vii) for each $i = 1, \ldots, t$, the generalized eigenspaces,

$$\mathbb{V}_{\lambda i}^{(r)} = K_{es}(T - \lambda_i Id)^r ,$$

$$\mathbb{V}_{\lambda i} = \mathbb{V}_{\lambda i}^{(1)} \subseteq \mathbb{V}_{\lambda i}^{(2)} \subseteq \ldots\ldots \subseteq \mathbb{V}_{\lambda i}^{(m_i)} = \mathbb{V}_{\lambda i}^{gen}$$ ← generalized eigenspace.

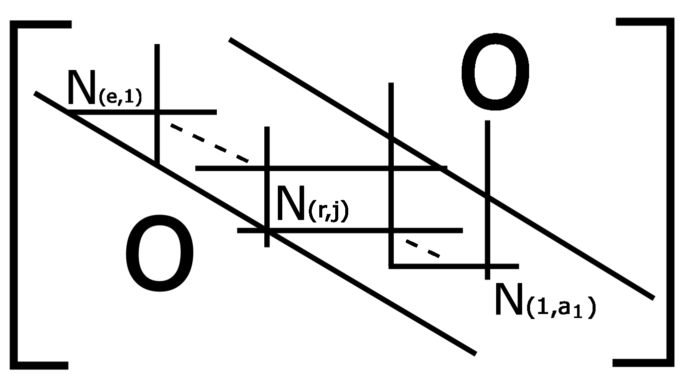
(viii) restriction of T to $\mathbb{V}_{\lambda i}^{gen}$ is T_i .

(ix) $N_i = T_i - \lambda_i Id_{\mathbb{V}_{\lambda i}^{gen}}$.

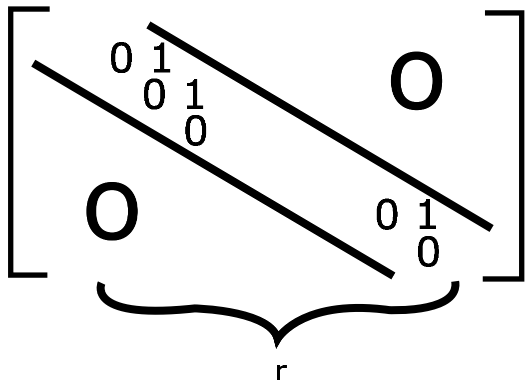
(x) Jordan normal form of N_i : we saw that there is a sequence of lin. ind. sets of vectors $B_{(e,1)}, \ldots, B_{(e,a_e)}, \ldots, B_{(r,1)}, \ldots, B_{(r,a_r)}, \ldots, B_{(1,1)} \ldots, B_{(1,a_1)}$ (of course not every integer r has to occur above)

such that $B_{\lambda i} = B_{(e,1)} \cup \ldots\ldots \cup B_{(1,a_r)}$ is an ordered basis for $\mathbb{V}_{\lambda i}^{gen}$ with respect to which we have

$[N_i]_{B_{\lambda i},\, B_{\lambda i}} =$



$N_{(r,j)}$ = r



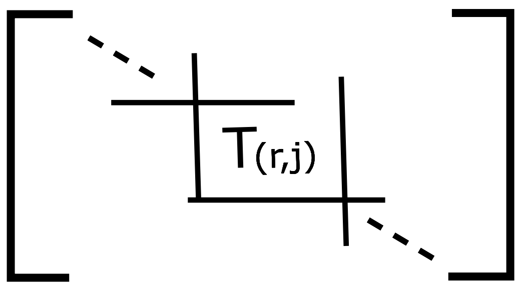
Gave usual method for finding the sets $B_{(r,j)}$:

First choose a maximal lin. ind. set of vectors $v_{(r,1)}, \ldots\ldots, v_{(r,a_r)}$ contained in

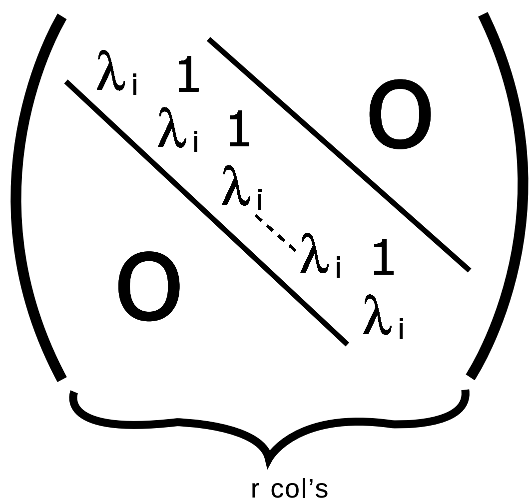
$\mathcal{V}_{\lambda i}^{(r)} - (N_i \mathcal{V}_{\lambda i}^{(r+1)} \cup \mathcal{V}_{\lambda i}^{(r-1)})$ and define

$B_{(r,j)} = (N_i^r \ v_{(r,j)}, N_i^{r-1} \ v_{(r,j)}, \ldots., N_i \ v_{(r,j)}, v_{(r,j)})$.

(xi) The Jordan normal form of T_i w.r.t B is



$T_{(r,j)}$ =



(xii) Restrict T to the subspace $\overline{W}_{(r,j)}$ spaced by $B_{(r,j)}$. Represent el'ts w.r.t basis

$B_{(r,j)}$ as $z = \begin{bmatrix} z_1 \\ z_2 \\ : \\ : \\ z_r \end{bmatrix}$. Then $z' = T_z$ is just

$$\begin{cases} z_1' = \lambda z_1 + z_2 \\ z_2' = \lambda z_2 + z_3 \\ \quad \vdots \\ \quad \vdots \\ z_{r-1}' = \lambda z_{r-1} + z_r \\ z_r' = \lambda z_r \end{cases}$$

(xiii) The solution space of $z' = T_z$ has a basis

$$z_{(l)} = \begin{bmatrix} t^l / l! \\ t^{l-1} / (l-1)! \\ \vdots \\ t/1 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} e^{\lambda_1 t}$$ for $l = 0, \ldots\ldots, r-1$, i.e.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} e^{\lambda_1 t}, \begin{bmatrix} t \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} e^{\lambda_1 t}, \begin{bmatrix} t^2/2 \\ t \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e^{\lambda_1 t}, \ldots\ldots\ldots$$

(xiv) In computing the Jordan normal form, we also find the matrix

$U = [Id_V]_{B^{old}, B^{new}}$, i.e. for the j^{th} basis vector v_j in the new basis,

$v_j = \sum_{i=1}^{n} u_{ij} e_i$, where ($e_1, \ldots, e_n$) is the original basis.

(xv) For each vector from (xiii) extend the column vector by zesos to get a solution $z^{(r,j,e)}$ of $z' = [T]_{B^{new}, B^{new}} z$.

The corresponding solution of the linear system w.r.t. the original basis is

$$\boxed{y_{(r,j,l)} = U z_{(r,j,l)}}$$

II. Lecture 30, part 1 $\approx$ (30 mins)

10mins A. An example with recil eigenvalues.

$$y' = Ay, \qquad A = \begin{bmatrix} 6 & -4 & 1 \\ 4 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = (e_1, e_2, e_3)$$

$P_A(\lambda) = (\lambda - 2)^3$. So one eigenvalue $\lambda = 2$ w/multiplicity 3.

$$[N]_{B,B} = A - 2I = \begin{bmatrix} 4 & -4 & 1 \\ 4 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[N^2]_{B,B} = (A - 2I)^2 = \begin{bmatrix} 0 & 0 & -8 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}, \quad [N^3]_{B,B} = 0 \text{ matrix}$$

$$V_2^{(1)} = K_{er}(N) = S_{pn}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right), \qquad V_2^{(2)} = S_{pn}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$\mathbb{V}_2^{(3)} = \mathbb{V} \ .$$

Choose $v_{(3,1)} = e_3$. Then $B' = B_{(3,1)} = \left(N^2 e_3, Ne_3, e_3\right)$

$$= \left(\begin{bmatrix} -8 \\ -8 \\ 0 \end{bmatrix} \text{N G ,} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \text{N G ,} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \qquad U = \begin{bmatrix} -8 & 1 & 0 \\ -8 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[N]_{B',B'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [T]_{B',B'} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

w.r.t. B' a basis for the solution space is

$$z_{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} , \qquad z_{(2)} = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} e^{2t} , \qquad z_{(3)} = \begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix} e^{2t}$$

w.r.t original basis B, a basis for the solution space is

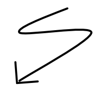
$$y_{(1)} = Uz_{(1)} = \begin{bmatrix} -8 \\ -8 \\ 0 \end{bmatrix} e^{2t} , \qquad y_{(2)} = Uz_{(2)} = \begin{bmatrix} -8t+1 \\ -8t+3 \\ 0 \end{bmatrix} e^{2t} , \; y_{(3)} = Uz_{(3)} = \begin{bmatrix} -4t^2+t \\ -4t^2+3t \\ 1 \end{bmatrix} e^{2t}$$

So the general real solution of $y' = Ay$ is

$$\boxed{y = C_1 \begin{bmatrix} -8 \\ -8 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} -8t+1 \\ -8t+3 \\ 0 \end{bmatrix} e^{2t} + C_3 \begin{bmatrix} -4t^2+t \\ -4t^2+3t \\ 1 \end{bmatrix} e^{2t}}$$

5 mins B. What if some eigenvalues are complex:

Answer : Go through the same process as before to get $y_{(r,j,l)} = Uz_{(r,j,l)}$. Then take the real and imaginary parts of $y_{(r,j,l)}$. You only need to do this for one of the two complex conjugate eigenvalues in each conjugate eigenvalue pair.



: It is not okay to take the real and imaginary parts of $z_{(r,j,l)}$!

The matrix U will have complex entries, and you need to first compute $z_{(r,j,l)}$.

10 mins-15 mins C. An example with complex eigen values

$\mathbb{V} = \mathbb{C}^4$, $B = (e_1, e_2, e_3, e_4)$. $T = T_A$,

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ -1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad P_A(\lambda) = \left((\lambda-2)^2+1\right)^2$$

$$\lambda_t = 2+i, \lambda_- = 2-i$$

$$\lambda_t : [N_t]_{B,B} = A - \lambda_t I = \begin{bmatrix} -i & 1 & -1 & 3 \\ -1 & -i & 0 & -2 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix},$$

$$[N_t^2]_{B,B} = \begin{bmatrix} -i & 1 & -3+2i & -3-6i \\ -1 & -i & 3 & -3+4i \\ 0 & 0 & -i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix}$$

The matrix $[N_t^2]_{B,B}$ is row equivalent to

$$\begin{bmatrix} 1 & i & 0 & -3-i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{So } \mathrm{V}_{\lambda_t}^{(1)} = S_{pn} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix},$$

$$\mathrm{V}_{\lambda_t}^{(2)} = S_{pn} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3-i \\ 0 \\ i \\ -1 \end{bmatrix}.$$

Choose $V_{(2,1)} = \begin{bmatrix} 3-i \\ 0 \\ i \\ -1 \end{bmatrix}$, $B_{(2,1)} = \left(\begin{bmatrix} -4-4i \\ 4-4i \\ 0 \\ 0 \end{bmatrix} \text{NG}, \begin{bmatrix} 3-i \\ 0 \\ i \\ -1 \end{bmatrix} \right)$.

With respect to this lin ind. set, T has matrix $\begin{bmatrix} 2+i & 1 \\ 0 & 2+i \end{bmatrix}$.

So the solution space for λ has a basis

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} e^{it}$, $\begin{bmatrix} t \\ 0 \end{bmatrix} e^{2t} e^{it}$

So $y_{(2,1,1)} = \begin{bmatrix} -4-4i \\ 4-4i \\ 0 \\ 0 \end{bmatrix} e^{2t} e^{it} = e^{2t} \begin{bmatrix} -4\cos(t) + 4\sin(t) \\ 4\cos(t) + 4\sin(t) \\ 0 \\ 0 \end{bmatrix} + i\, e^{2t} \begin{bmatrix} -4\cos(t) - 4\sin(t) \\ -4\cos(t) + 4\sin(t) \\ 0 \\ 0 \end{bmatrix}$.

And $y_{(2,1,2)} = \begin{bmatrix} 3 - i + t(-4-4i) \\ t(4-4i) \\ i \\ -1 \end{bmatrix} e^{2t} e^{it}$

$$y_{(2,1,2)} = e^{2t}\begin{bmatrix} t(-4\cos(t)+4\sin(t))+3\cos(t)+\sin(t) \\ t(4\cos(t)+4\sin(t)) \\ -\sin(t) \\ -\cos(t) \end{bmatrix} + i e^{2t}\begin{bmatrix} t(-4\cos(t)-4\sin(t))-\cos(t)+3\sin(t) \\ t(-4\cos(t)+4\sin(t)) \\ \cos(t) \\ -\sin(t) \end{bmatrix}$$

So a basis for the solution space is

$$y_{(1)} = \begin{bmatrix} -4\cos(t)+4\sin(t) \\ 4\cos(t)+4\sin(t) \\ 0 \\ 0 \end{bmatrix} e^{2t}, \quad y_{(2)} = \begin{bmatrix} -4\cos(t)-4\sin(t) \\ -4\cos(t)+4\sin(t) \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

$$y_{(3)} = \begin{bmatrix} t(-4\cos(t)+4\sin(t))+3\cos(t)+\sin(t) \\ t(4\cos(t)+4\sin(t)) \\ -\sin(t) \\ -\cos(t) \end{bmatrix} e^{2t},$$

$$y_{(4)} = \begin{bmatrix} t(-4\cos(t)-4\sin(t))-\cos(t)+3\sin(t) \\ t(-4\cos(t)+4\sin(t)) \\ \sin(t) \\ -\cos(t) \end{bmatrix} e^{2t}$$

III. Lecture 30, part 2 ($\approx$20 min's)
Discuss the matrix exponential and its use to solve inhomogeneous linear systems with constant coeffs: § 6,6 of Borelli + Coleman.
Of course, the books definition of the matrix exponential is ridiculous. Also, both for finding solutions of inhomog. systems and for solving IVP's, it is better to go through the process above and then compute the matrix $U^{-1} = [Id]_{B^{new},B^{old}}$.

Then for an IVP $\begin{cases} y' = Ay + f \\ y(0) = y_0 \end{cases}$, we get

the transformed IVP

$$\begin{cases} z' = U^{-1}AUz + U^{-1}f \\ z(0) = z_0 = U^{-1}y_0 \end{cases}$$

Because $B = U^{-1}AU$ is in Jordan normal form, it is usually simpler to solve this directly than to compute the matrix exponential.