18.034 Honors Differential Equations
Spring 2009

UNIT III: HIGHER-ORDER LINEAR EQUATIONS

We give a comprehensive development of the theory of linear differential equations with constant coefficients. We use the operator calculus to deduce the existence and uniqueness. We presents techniques for finding a complete solution of the inhomogeneous equation from solutions of the homogeneous equation. We also give qualitative results on asymptotic stability.

LECTURE 11. HIGHER-ORDER LINEAR EQUATIONS

The n-th order linear differential equation with constant coefficient is

$$Ly = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = f(t), \tag{11.1}$$

where $y^{(k)} = \dfrac{d^k y}{dt^k}$ is the k-th derivative of y with respect to t, and p_j are real or complex constants, $f(t)$ is a continuous function on an interval I. The letter L stands for the (homogeneous) differential operator. It is easy to see that $L : C^n(I) \to C(I)$ is linear, where $C^k(I)$ is the space of functions differentiable k times on I.

As for the second-order equations treated in Unit II, the *principle of superposition* and the *principle of the complementary solution* apply to (11.1).

Principle of Superposition. If $Lu = 0$ and $Lv = 0$, where L is given in (11.1), then $L(c_1u + c_2v) = 0$ for any constants c_1 and c_2.

Principle of the Complementary Solution. If u is a particular solution of $Lu = f$, where L is given in (11.1), and if v is any solution of $Lv = 0$, then $L(u + v) = f$ and every solution of $Ly = f$ can be obtained this way.

Therefore, the general solution of (11.1) is given as

$$y = y_p + y_h,$$

where y_p is a particular solution of (11.1) and y_h is a solution of the corresponding homogeneous equation

$$Ly = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = 0. \tag{11.2}$$

The characteristic polynomial. We try $y(t) = e^{\lambda t}$, $\lambda \in \mathbb{C}$, as a solution of the homogeneous equation (11.2). Since $\dfrac{d^k}{dt^k}(e^{\lambda t}) = \lambda^k e^{\lambda t}$, the substitution yields

$$Le^{\lambda t} = (\lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n)e^{\lambda t} = 0.$$

Moreover, since $e^{\lambda t}$ is never zero, $Le^{\lambda t} = 0$ if and only if λ is a root of the *characteristic polynomial*

$$p_L(\lambda) = \lambda^n + p_1\lambda^{n-1} + ... + p_{n-1}\lambda + p_n. \tag{11.3}$$

of L.

Example 11.1. We recall the results for the second-order equation

$$Ly = y'' + py' + qy, \tag{11.4}$$

where p , q are constants. The roots of the characteristic polynomial are

$$\lambda = \frac{-p \pm \sqrt{\Delta}}{2}, \qquad \Delta = p^2 - 4q.$$

If $p^2 > 4q$ then $y = e^{\lambda t}$ with the above λ are solutions of $Ly = 0$.

We will show how the exponential substitution $y = e^{\lambda t}$ applies to solve the general differential equations of all order (11.2).

The operator calculus. The study of linear differential equations become easier if we introduce an abstract symbol $D = d/dt$ for the operation of differentiation. A word of caution. The symbol d is used for differentials. e.g., $D(t^3) = 3t^2$, but $d(t^3) = 3t^2 dt$.

As an operator, D is linear. That is,

$$D(u + v) = Du + Dv, \qquad D(cu) = cDu$$

for all differentiable functions u, v and for any constant c.

We now list some properties of D. By definition,

$$D^0 = id, \qquad D^k = \frac{d^k}{dt^k} \quad k = 1, 2, \ldots.$$

Moreover,

$$D^j D^k = D^{j+k}, \qquad (D^j)^k = D^{jk}, \qquad j, k = 1, 2, \ldots. \tag{11.5}$$

The proof is left as an exercise.

With the notation D, we may write the differential operator in (11.2) as

$$Ly = (D^n + p_1 D^{n-1} + \cdots + p_{n-1} D + p_n)y.$$

Then, it can be recognized that the first factor of the right side is $p_L(D)$, the characteristic polynomial p_L evaluated, formally, at D. In this sense, we say $L = p_L(D)$.

Linear operators with constant coefficients are *permutable*, in the sense of the following.

Lemma 11.2. *If $p(D) = \sum a_j D^j$ and $q(D) = \sum b_k D^k$ are two linear differential operator, where a_j , b_k are constants. then,*

$$p(D)q(D) = q(D)p(D) = \sum a_j b_k D^{j+k}.$$

The proof uses (11.5) and it is left as an exercise.

Remark 11.3. The above lemma is *not* true of linear operators with variable coefficients. For example,

$$D(tf) = (tf)' = tf' + f = (tD + id)f,$$

where f is a differentiable function of t. In other words, $Dt = tD + id$.

In many applications, one takes trial solutions of the form $e^\lambda tu$ where $\lambda \in \mathbb{C}$ and u is a function with a certain degree of smoothness. Thus, it is useful to know how such a function works with the operator D.

Lemma 11.4 (The Exponential Shift Laws). *If p is a polynomial and λ is a constant, then*

$$p(D)(e^{\lambda t} f) = e^{\lambda t} p(D + \lambda) f.$$

Proof. By the rule for differentiation,

$$D(e^{\lambda t}f) = e^{\lambda t}Df + \lambda e^{\lambda t} = e^{\lambda t}(D+\lambda)f.$$

Hence,

$$(D-a)(e^{\lambda t}f) = e^{\lambda t}(D-a+\lambda)f$$

for any constant a. Then, by induction,

$$(D-a)^k(e^{\lambda t}f) = e^{\lambda t}(D-a+\lambda)^k f. \tag{11.6}$$

Finally, by the fundamental theorem of calculus, any polynomial p can be factorized as

$$p(D) = (D-a_1)^{k_1}(D-a_2)^{k_2}\cdots(D-a_m)^{k_m},$$

where $a_j \in \mathbb{C}$ are roots of the polynomial and $k_j \geqslant 1$ are the corresponding multiplicity. The assertion then follows by Lemma 11.2 and (11.6). □

As a consequence, moreover, we have

$$(D-\lambda)(e^{\lambda t}f) = e^{\lambda t}Df, \qquad (D-\lambda)^k(e^{\lambda t}f) = e^{\lambda t}D^k f.$$

Exercise. If a is not a root of the polynomial p , then show that

$$b(t) = \frac{e^{at}}{p(a)}$$

is a particular solution of the differential equation $p(D)y = e^{at}$.

We now present our main result.

Theorem 11.5. *If λ_* is a (complex) root of multiplicity k of the characteristic polynomial $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ of the linear differential operator $p(D)$ with constant coefficients, then the functions $t^r e^{\lambda_* t}$, where $r = 0, 1, \ldots, k-1$, are solution of $p(D)y = 0$.*

Proof. By the exponential shift law, it follows that

$$(D-\lambda_*)^k(t^r e^{\lambda_* t}) = e^{\lambda_* t}D^k t^r = 0$$

for $r = 0, 1, \ldots, k-1$.

On the other hand, $p(\lambda)$ must contain the factor $(\lambda-\lambda_*)^k$, and hence

$$p(D) = (D-\lambda_*)^k q(D), \qquad q(D) = \prod_{\lambda_j \neq \lambda}(D-\lambda_j)^{k_j}.$$

Finally, by Lemma 11.2

$$p(D)(t^r e^{\lambda_* t}) = q(D)(D-\lambda_*)^k(t^r e^{\lambda_* t}) = 0.$$

This completes the proof. □

Corollary 11.6. *If*

$$p(\lambda) = (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdot(\lambda-\lambda_m)^{k_m}$$

then the functions $t^r e^{\lambda_j t}$, where $r = 0, 1, 2, \ldots, \lambda_{j-1}$ and $j = 1, 2, \ldots, m$, are solutions of the differential equation $p(D)y = 0$.

Complex solutions. An interesting feature of the analysis via the operator calculus is that many problems are best solved by the use of complex-valued functions, even when the coefficients are real, e.g. (11.4) when $p^2 < 4q$.

Theorem 11.5 holds whether the coefficients p_j of $p(D)y = 0$ are real or complex. In fact, although t is interpreted as real (in particular in the discussion of stability) the operator calculus and the solutions constructed with it apply equally well to functions of the complex variable. But, when the coefficients p_j are real, then we may obtain a stronger conclusion.

We recall that the complex roots of a polynomial come in pair, $\mu \pm i\nu$, $\mu, \nu \in \mathbb{R}$. We also recall the complex exponential $e^{\mu \pm i\nu} = e^{\mu}(\cos\nu \pm i\nu)$.

Lemma 11.7 (Principle of Equating Real Parts)**.** *If a complex-valued function $y(t) = u(t) + iv(t)$, where u and v are real-valued functions, satisfies the differential equation* (11.2) *with real coefficients, then $u(t)$ and $v(t)$, the real and imaginary parts of y, both satisfy* (11.2).

Proof. Let $Ly = L(u + iv) = 0$. Since the coefficients of L are real, taking the complex conjugate we have $\overline{Ly} = L(u - iv) = 0$. Then, by linearity,

$$u = \frac{y + \overline{y}}{2} \quad \text{and} \quad v = \frac{y - \overline{y}}{2i}$$

both satisfy (11.2). □

It holds true for differential equations with real-variable coefficients.

Corollary 11.8. *Each pair of complex roots $\mu \pm i\nu$, $\mu, \nu \in \mathbb{R}$, of multiplicity k of the polynomial p gives real solutions $t^r e^{ut} \cos\nu t$, $t^r e^{ut} \sin\nu t$, where $r = 0, 1, \ldots, k - 1$, of the differential equation $p(D)y = 0$.*