18.034 Honors Differential Equations
Spring 2009

LECTURE 12. SOLUTION BASES

We present the results pertaining to the linear differential equation

$$L_0 y = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y' + p_n y \tag{12.1}$$

with constant coefficients. Some results we establish apply to equations with variable coefficients.

Let $p(\lambda) = \lambda^n + p_1 \lambda^{(n-1)} + \cdots + p_{n-1}\lambda + p_n$ be the characteristic polynomial corresponding to L_0. By the fundamental theorem of algebra*, p is factored into linear factors in the complex fields as

$$p(\lambda) = (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_m)^{k_m},$$

where λ_j are (complex) roots of $p(\lambda)$ and $k_j \geqslant 1$ are the multiplicity of λ_j.

We recall from the last lecture that the functions

$$t^r e^{\lambda_j t}, \qquad \text{where} \quad r = 0, 1, \cdots, \lambda_{j-1}, \quad \text{and} \quad j = 1, 2, \cdots, m,$$

are (complex) solutions of $L_0 y = 0$. Moreover, each pair of complex conjugate roots, $\lambda_j = \mu + i\nu_j$, $\bar{\lambda} = \mu_j - i\nu_j$, with $\mu_j, \nu_j \in \mathbb{R}$, gives real solutions $t^r e^{\mu_j t}\cos \nu_j t$, $t^r e^{\mu_j t} \sin \nu_j t$, where $r = 0, 1, \cdots, k_{j-1}$, of $L_0 y = 0$.

Our goal is to show that all solutions of $L_0 y = 0$ are linear combinations of these n solutions. Namely, these solutions form a *basis* of solutions of $L_0 y = 0$.

Linear independence. There are two notions of linear independence, depending on the scalar field. A set os n real or complex functions $f_1, \cdots, f_n$ defined on an interval I is said linearly independent *over the real field* if

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t) = 0 \quad \text{on } I,\ c_j \in \mathbb{R}, \qquad \text{implies} \qquad c_j = 0 \quad \text{for all } j.$$

We may define, similarly, a set of real or complex functions to be linearly independent *over the complex field*.

Lemma 12.1. *A set of real-valued functions on an interval I is linearly independent over the real field if and only if it is linearly independent over the complex field.*

Next, we state the main result of this subsection.

Lemma 12.2. *Any set of functions of the form*

$$f_{rj}(t) = t^r e^{\lambda_j t}, \qquad j = 1, 2, \cdots, n, \tag{12.2}$$

where the r are nonnegative integers and $\lambda_j \in \mathbb{C}$, is linearly independent on any nonempty open interval, unless two or more of the functions are identical.

Proof. Suppose that $f_{rj}(t)$ are all distinct. Suppose that

$$\sum c_{rj} f_{rj}(t) = 0 \text{ on an open interval of } t, \text{but } c_{rj} \neq 0 \text{ for some r and } j.$$

Fix such a j, and choose R to be the largest r such that $c_{rj} \neq 0$.

We form a linear differential operator of constant coefficient

$$p(D) = (D - \lambda_j)^R \prod_{i \neq j} (D - \lambda_i)^{k_i + 1},$$

*It was first proved by Carl Friedrich Gauss.

where k_i is the largest r such that $t^r e^{\lambda_i t}$ belongs to te set of functions in (12.2). It is obvious that $p(D)(\sum c_{rj} f_{rj}) = p(D)(0) = 0$.

On the other hand,

$$\begin{aligned} p(D)(\sum c_{rj} f_{rj}) =& \prod_{i \neq j} (D - \lambda_i)^{k_i+1} (D - \lambda_j)^R (\sum c_{rj} f_{rj}) \\ =& c_{Rj} \prod_{i \neq j} (D - \lambda_i)^{k_i+1} (D - \lambda_j)^R (t^R e^{\lambda_j t}) \\ =& c_{Rj}(R!) \prod_{i \neq j} (\lambda_j - \lambda_i)^{k_i+1} e^{\lambda_i t} \neq 0. \end{aligned}$$

A contradiction then proves the theorem. □

Corollary 12.3. *The differential equation $L_0 y = 0$, where L_0 is given in* (12.1), *has at least n linearly independent, real or complex solutions of the form $t^r e^{\lambda t}$.*

The differential equation $L_0 y = 0$, where p_j are real constants, has a set of n solutions of the form

$$t^r e^{\lambda t}, \quad \textit{or} \quad t^r e^{\mu t} \cos \nu t, \quad t^r e^{\mu t} \sin \nu t$$

which is linearly independent over the real field in any nonempty interval.

Solution bases. We now show that all solutions of the homogeneous equation $L_0 y = 0$, where L_0 is given in (12.1) and p_j are real, are linear combinations of the special solutions obtained in Corollary 12.3. To this end, we establish some results for the more general equation

(12.3) $$Ly = y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = 0,$$

where $p_j(t)$ are real-valued continuous functions on an interval I.

The *solution set*, denoted by $N(L)$, of $Ly = 0$ is a collection of solutions of $Ly = 0$.

Exercise. Show that $N(L)$ forms a linear subspace of $C^n(I)$.

A *basis of solutions* of $Ly = 0$ is then defined as a basis of $N(L)$, as a linear space. In other words, any solution of $Ly = 0$ is uniquely expressed as a linear combination of members in the basis of solutions.

We associate to the differential equation $Ly = 0$, where L is given in (12.3), a transformation

$$T : N(L) \to \mathbb{R}^n, \qquad Ty = (y(t_0), y'(t_0), \cdots, y^{n-1}(t_0)).$$

It is clear that T is linear. We show that T is one-to-one. That is, $Ly = 0$ has uniqueness.

Lemma 12.4. *(Uniqueness) If y is a real or complex solution of $Ly = 0$, where L is given in* (12.3) *and p_j are real-valuead continuous in the closed interval I containing t_0, and if*

$$y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0,$$

then $y(t) = 0$ for all $t \in I$.

Proof. The proof is similar to that for the second-order equations. If y is a real solution, then let

$$E(t) = y^2(t) + (y'(t))^2 + \cdots + (y^{(n-1)}(t))^2$$

and derive a differential inequality $\dfrac{dE}{dt} \leqslant KE$ for some constant $K > 0$. The detail is left as an exercise. If y is a complex solution, then its real and imaginary parts are both real solutions of $Ly = 0$ satisfying the initial conditions. This completes the proof. □

Furthermore, if we show that T is onto, that is, for any n-vector $(y_0, y_1, \cdots, y_{n-1}) \in \mathbb{R}^n$, the differential equation $Ly = 0$ has a solution satisfying the initial conditions

$$y(t_0) = y_0, \cdots, y^{(n-1)}(t_0) = y_{n-1},$$

then it implies that T is an isomorphism. In particular, the dimension of $N(L)$ is n. Thus, a basis of solutions of $Ly = 0$ consists of n functions.

If the coefficients of L are real constants, that is, $L = L_0$ where L_0 is in (12.1), then $Ly = 0$ has a n linearly independent (real) solutions of the form

$$t^r e^{\lambda t}, \qquad t^r e^{\mu t} \cos \nu t, \qquad t^r e^{\mu t} \sin \nu t.$$

In this case, these n solutions form a basis of solutions of $Ly = 0$. This result holds true for equations with real-variable coefficients.

Exercise. If $y_1, \cdots, y_n$ are n linearly independent solutions of $Ly = 0$, where L is given in (12.3) and $p_j(t)$ are real-valued contiuous on an interval $t_0 \in I$, then show that for given arbitrary real numbers $y_0, y_1, \cdots, y_{n-1}$ there exist unique constant $c_1, \cdots, c_n$ such that $y(t) = \sum_{j=1}^{n} c_j y_j(t)$ is a solution of $Ly = 0$ satisfying

$$y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{n-1}(t_0) = y_{n-1}.$$

In other words, $y_1, \cdots, y_n$ form a basis of solutions of $Ly = 0$.

The Euler-Cauchy equation. There are only a few interesting classes of n-th order linear equations for which a basis of solutions can be expressed in terms of elementary functions. One is the class of equations with constant coefficients, which we discussed. Another is the class of equations of the form

$$x^n y^{(n)} + p_1 x^{n-1} y^{(n-1)} + \cdots + p_{n-1} x y' + p_n y = 0, \qquad x > 0$$

where $y^k = \dfrac{d^k y}{dx^k}$ is the k-th derivative with respect to x and p_j are constants. An equation of this form is called *Cauchy's equi-dimensional equation*, although equations of this kind were studied earlier by Euler.

The substitution $x = e^t$ leads to an equation with constant coefficients and a basis of solutions consists of functions of the form

$$t^r e^{\lambda t}, \qquad t^r e^{\mu t} \cos \nu t, \qquad t^r e^{\mu t} \sin \nu t.$$

In terms of x, then

$$x^\lambda (\log x)^r, \qquad x^\mu \cos(\nu \log x), \qquad x^\mu \sin(\nu \log x).$$