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LECTURE 26. EIGENVALUES AND EIGENVECTORS

We study the system

$$\vec{y}\,' = A\vec{y}, \tag{26.1}$$

where $A = (a_{ij})$ is a constant $n \times n$ matrix.

When $n = 1$, the above system reduces to the scalar equation $y' = ay$, and it has solutions of the form ce^{at}. For $n \geqslant 2$, similarly, we try solutions of the form $\vec{v}e^{\lambda t}$, where $\vec{v} \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$. Then, (26.1) becomes

$$\lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t}.$$

Subsequently,

$$A\vec{v} = \lambda\vec{v}, \qquad (A - \lambda I)\vec{v} = 0. \tag{26.2}$$

In order to find a solution (26.1) in the form of $\vec{v}e^{\lambda t}$ we want to find a nonzero vector $\vec{v}$ and $\lambda \in \mathbb{C}$ satisfying (26.2). It leads to the following useful notions in linear algebra.

Definition 26.1. A nonzero vector $\vec{v}$ satisfying (26.2) is called an *eigenvector* of A with the *eigenvalue* $\lambda (\in \mathbb{C})$.

These words are hybrids of English and German, and they follow German usage, "ei" rhymes with π.

We recognize that (26.2) is a linear system of equations for $\vec{v}$. A well-known result from linear algebra is that it has a nontrivial solution $\vec{v}$ if and only if $A - \lambda I$ is singular. That is, $p_A(\lambda) = |A - \lambda I| = 0$, where $p_A(\lambda)$ is the characteristic polynomial of A. In this case, such a nontrivial solution $\vec{v}$ is an eigenvector and the corresponding root of $p_A(\lambda) = 0$ is an eigenvalue.

Plane systems. For a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (\text{tr}\, A)\lambda + \det A.$$

This quadratic polynomial has two roots, λ_1 and λ_2 (not necessarily distinct). Let $\vec{v}_1$ and $\vec{v}_2$ be the eigenvectors corresponding to the eigenvalues λ_1 and λ_2, respectively. By definition, $\vec{v}_1e^{\lambda_1 t}$ and $\vec{v}_2e^{\lambda_2 t}$ are solutions of (26.1).

If $\lambda_1 \neq \lambda_2$, then the functions $\vec{v}_1e^{\lambda_1 t}$ and $\vec{v}_2e^{\lambda_2 t}$ are linearly independent. Hence, they form a basis of solutions of (26.1).The general solution of (26.1) is given as

$$\vec{y}(t) = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t},$$

where c_1, c_2 are arbitrary constants. This shows one use of eigenvalues in the study of (26.1).

Let us define 2×2 matrices

$$V = (\vec{v}_1 \quad \vec{v}_2) \quad \text{and} \quad \Lambda = \text{diag}\,(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

One can verify that $AV = V\Lambda$. If $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, so that $|V| \neq 0$, then we can make the (non-singular) change of variables

$$\vec{x} = V^{-1}\vec{y}.$$

Then, (26.1) is transformed into $\vec{x}\,' = \Lambda\vec{x}$, that is

$$x_1' = \lambda_1 x_1,$$
$$x_2' = \lambda_2 x_2.$$

That is, $\vec{x}$ solves a decoupled system. The solution of this system is immediate and $x_1 = c_1 e^{\lambda_1 t}$, $x_2 = c_2 e^{\lambda_2 t}$. The new variables $\vec{x}$ is called the *canonical variables*, and $\Lambda = V^{-1}AV$ is called the *diagonalization*. This is another use of eigenvalues. Canonical variables play a major role i engineering, economics, mechanics, and indeed in all fields that makes intensive use of linear systems with constant coefficients.

Lemma 26.2. *If the eigenvalues of a 2×2 matrix are distinct, then the corresponding eigenvectors are linearly independent.*

Proof. Suppose that the eigenvalues $\lambda_1 \neq \lambda_2$, but the eignevectors satisfy

$$c_1\vec{v}_1 + c_2\vec{v}_2 = 0. \tag{26.3}$$

We want to show that $c_1 = c_2 = 0$. Applying the matrix A to (26.3), we obtain that

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = 0. \tag{26.4}$$

Subtraction then yields

$$c_1(\lambda_2 - \lambda_1)\vec{v}_1 = 0.$$

This implies $c_1 = 0$. Then, (26.3) implies $c_2 = 0$.

□

Example 26.3. We consider $A = \begin{pmatrix} 12 & 5 \\ -6 & 1 \end{pmatrix}$.

Its characteristic polynomial is $p(\lambda) = \lambda^2 - 13\lambda + 42 = (\lambda - 6)(\lambda - 7)$, and A has two distinct eigenvalues, $\lambda_1 = 6$ and $\lambda_2 = 7$.

If $\lambda_1 = 6$, then $A - 6I = \begin{pmatrix} 6 & 5 \\ -6 & -5 \end{pmatrix}$, and $\begin{pmatrix} 5 \\ -6 \end{pmatrix}$ is an eigenvector.

If $\lambda_2 = 7$, then $A - 7I = \begin{pmatrix} 5 & 5 \\ -6 & -6 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector. The general solution of (26.1) is, thus,

$$y(t) = c_1 e^{6t}\begin{pmatrix} 5 \\ -6 \end{pmatrix} + c_2 e^{7t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The canonical variable is $\vec{x} = \begin{pmatrix} 5 & 1 \\ -6 & -1 \end{pmatrix}^{-1}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 - y_2 \\ 6y_1 + 5y_2 \end{pmatrix}$.

Exercises. 1. Show that $A = \begin{pmatrix} 6 & 1 \\ -1 & 8 \end{pmatrix}$ has only one eigenvalue $\lambda = 7$, and the only corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In this case, we can't construct the general solution of (26.1) from this.

2. Show that $A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ has two (complex) eigenvalues $1 \pm 2i$ and the corresponding eigenvectors $\begin{pmatrix} 1 \\ \pm 2i \end{pmatrix}$, respectively. They leads to the general (complex) solution of (26.1)

$$c_1 e^{(1+2i)t}\begin{pmatrix} 1 \\ 2i \end{pmatrix} + c_2 e^{(1-2i)t}\begin{pmatrix} 1 \\ -2i \end{pmatrix}.$$

Higher-dimensional systems. If A is an $n \times n$ matrix, where $n \geqslant 1$ is an integer, then

$$p_A(\lambda) = |A - \lambda I|$$

is a polynomial in λ of degree n and has n roots, not necessarily distinct. That means, A has n eigenvalues $\lambda_1, \dots, \lambda_n$, not necessarily distinct. Let $\vec{v}_1, \dots, \vec{v}_n$ be the eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$, respectively. Let

$$V = (\vec{v}_1 \ \cdots \ \vec{v}_n) \quad \text{and} \quad \Lambda = \text{diag}\,(\lambda_1, \dots, \lambda_n).$$

By definition, $\vec{v}_1 e^{\lambda_1 t}, \dots, \vec{v}_n e^{\lambda_n t}$ are solutions of (26.1).

If $|V| \neq 0$, that means, if $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent, then they form a basis of solutions of (26.1). Moreover, $\vec{x} = V^{-1}\vec{y}$ is a canonical variable and $\vec{x}\,' = \Lambda\vec{x}$. In many cases, the vectors $\vec{v}_j$ can be chosen linearly independent even if λ_j are not all distinct. Sometimes the condition $|V| \neq 0$ is met by the following.

Lemma 26.4. *If eigenvalues $\lambda_1, \dots, \lambda_n$ are distinct, then the corresponding eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.*

Proof. Suppose not. Let $m > 1$ be the minimal number of vectors that are linearly dependent. Without loss of generality, we assume that $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent, so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0 \tag{26.5}$$

and some c_j is nonzero. We further assume that $c_2 \neq 0$.

We now proceed similarly to Lemma 26.2. Applying A to (26.5) we obtain

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_m\lambda_m\vec{v}_m = 0.$$

Multiplying by λ_1 then

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 + \cdots + c_m\lambda_1\vec{v}_m = 0.$$

Thus, we have

$$c_2(\lambda_2 - \lambda_1)\vec{v}_2 + \cdots + c_m(\lambda_m - \lambda_1)\vec{v}_m = 0.$$

Since $\vec{v}_2, \dots, \vec{v}_m$ are linearly independent, $c_2 = \cdots = c_m = 0$ must hold. A contradiction then proves the assertion. $\square$

We recall that if $A = A^T$ then the square matrix A is called *symmetric*. If a complex matrix satisfies $A = A^*$, where A^* denotes the conjugate transpose or adjoint of A, then A is called *Hermitian*. A symmetric or a Hermitian matrix has many important properties pertaining to the study of (26.1) via eigenvalues.

(1) All eigenvalues of a symmetric matrix are real and eigenvalues of A corresponding to different eignevalues are orthogonal.

Proof. Let

$$A\vec{u} = \lambda\vec{u}, \qquad A\vec{v} = \nu\vec{v},$$

$\vec{u}, \vec{v} \neq 0$ and $\lambda \neq \mu$. Then,

$$\vec{u}^T A\vec{u} = \lambda\vec{u}^T\vec{u}, \qquad \vec{u}^T A^T\vec{u} = \bar{\lambda}\vec{u}^T\vec{u}.$$

Since $A = A^T$, it implies that $\lambda = \bar{\lambda}$. The second assertion is left as an exercise. $\square$

(2) A has n linearly independent eigenvectors (regardless of the multiplicity of eigenvalues).

An immediate consequence of (1) and (2) is the following.

(3) If eigenvalues are simple (multiplicity $= 1$) then the corresponding eigenvectors are orthogonal.