

PROFESSOR: So today we're going to take a look at sinusoidal functions. And specifically, we're going to use complex numbers to get a handle on sinusoidal functions. The reason is complex numbers provide a robust way of analyzing sinusoidal functions.

So specifically, we're interested in this function, e to the $i\omega t$ divided by $2 + 3i$. And we're asked to write the real part of this function using polar form and then rectangular form. And then secondly, we're asked several properties of this function, the real part of this function. What's the circular frequency? What's the amplitude? And what's the phase lag? And then lastly, we're asked to sketch the real part of this function versus time.

So I'll let you take a look at this and try it for yourself. And I'll come back in a moment.

Hi everyone. Welcome back. OK. So let's take a look at this problem.

So we're asked to write e to $i\omega t$ divided by $2 + 3i$. We're asked to write the real part of this function, using polar form and then also rectangular form. So I'll first start off with polar form. And if we take a look at this function, we see that the numerator is already written in polar form. So just recall that polar form is of the form $r e$ to the $i\theta$.

Meanwhile, the denominator is written in Cartesian form, or rectangular form. So we need to convert this into polar form. So what we can do is we can write $2 + 3i$. We can combine these two into a modulus. And the modulus is going to be the square root of $2^2 + 3^2$. So this is the modulus of the complex number.

And then we have e to the $i\phi$. And the angle ϕ we can deduce by writing a triangle, which is over to the right by two units, and then it's up two units. And this is the angle ϕ . So we see from the triangle that the tangent of ϕ is equal to $3/2$.

OK. So putting the pieces together, we have e to the $i\omega t$ divided by $\sqrt{13} e$ to $i\phi$. And the beautiful thing about polar form is that when we divide and multiply these complex exponentials, it just turns into a simple addition or subtraction of the phases.

So we have $1/\sqrt{13}$, e to the $i\omega t - \phi$. So we've successfully combined all the terms into one polar term. And then now we're asked to compute the real part. And here we use Euler's formula. So just to recall Euler's formula, we have e to the $i\theta$. Then this is equal to $\cos \theta + i \sin \theta$. OK?

So at the end of the day, we're interested in the real part-- and I'm going to use this notation Re with a curly bracket to denote the real part-- of $1/\sqrt{13}$. And I'll use Euler's formula on $e^{i\omega t - \phi}$. So we have $\cos(\omega t - \phi) + i \sin(\omega t - \phi)$. Sorry. $1/\sqrt{13}$ is multiplying both the sine and the cosine.

And we see that the second term is purely imaginary. So when we take the real part of a real number plus an imaginary number, we just are left with the real number at the end of the day. So the answer we're looking for is $1/\sqrt{13} \cos(\omega t - \phi)$, where from before ϕ is the arctangent of $3/2$. And ϕ can also be thought of-- the angle being between zero and $\pi/2$.

OK. So this concludes the polar form computation. Secondly, we also have to compute this using a rectangular form calculation. So let's do the calculation in rectangular form. And for this, I'm going to use Euler's formula to expand out the numerator.

So first, we want to take the real part of the numerator is $\cos(\omega T) + i \sin(\omega t)$. The denominator was $2 + 3i$. And typically what we want to do is we want to turn the denominator into an entirely real number. So what we do is we multiply the top and bottom by the complex conjugate of the denominator.

So for example, I have $2 + 3i$. So I'm going to multiply the top and bottom by $2 - 3i$.

OK. And when I do this, you'll note that when we multiply out $2 + 3i$ and $2 - 3i$ we have a difference of squares. So we're left with: 2×2 is 4, the cross terms cancel. And then we have $3i$ times negative $3i$. That's going to give us negative $9i^2$, which is plus 9.

So the denominator is going to be $4 + 9$. And then the numerator-- I'll write out all the terms just for completeness-- it'll be $2 \cos(\omega t) + 3 \sin(\omega t)$. And then now we have some imaginary terms, which are negative $3 \cos(\omega t)$ and plus $2 \sin(\omega t)$.

OK?

But now we're going to take the real part of this complex function. And if we take a look at it, this first term is entirely real. This second term is entirely complex because we're multiplying by an i . So when we take the real part, at the end of the day, the second term is going to drop out.

So when the dust settles, we have $1/13 (2 \cos(\omega t) + 3 \sin(\omega t))$. OK? So

this concludes Part A.

For Part B, we're asked several questions about this sinusoidal function when written in its real form, specifically, what's the circular frequency. And the circular frequency is just the frequency that this function oscillates at. And we can get this from looking at the rectangular form or the polar form. And it's simply just going to be ω . So note how this function oscillates with frequency ω .

Secondly, we're asked about the amplitude. And if we take a look at it, the amplitude is a little tricky to get from the rectangular form, but it's completely apparent from the polar form. So if we take a look at the polar form, we have $\frac{1}{\sqrt{13}} \cos(\omega t - \phi)$.

And cosine oscillates between plus 1 and minus 1. So this function takes on a maximum value of plus $\frac{1}{\sqrt{13}}$ and a maximum negative value of negative $\frac{1}{\sqrt{13}}$. So the amplitude of this function is just going to be $\frac{1}{\sqrt{13}}$.

And then lastly, the phase lag. This is the angle that the cosine is picked up and shifted over. So again, we can get this directly from the polar form.

Meanwhile, it's a little less obvious what this angle is when we take a look at the rectangular form. So in our case, the phase lag is just going to be ϕ , which is $\tan^{-1} \frac{3}{2}$. OK? So this concludes Part B.

And then lastly, for Part C we're just asked to sketch this diagram. And again, the most direct way at sketching this function is through the polar form. So I'll write time on this axis. And if we just recall that a cosine might look something like this. So this is a cosine function.

And in our case, we're looking at plotting $\frac{1}{\sqrt{13}} \cos(\omega t - \phi)$. So we're interested in a cosine which has an amplitude of $\frac{1}{\sqrt{13}}$ and a maximum negative value of $-\frac{1}{\sqrt{13}}$ whose frequency is ω . So there's a quick way on finding zeros. So if the zero of a regular cosine is at $\frac{\pi}{2}$, then if we have an angular frequency of ω , the zero here is going to be $\frac{\pi}{2\omega}$. And this zero is going to be at $\frac{3\pi}{2\omega}$.

And lastly, this is not necessarily the cosine of our function. Our function is this cosine. However it must be picked up and shifted over by some phase lag ϕ . So in our case, the function we're actually looking for is some shifted cosine, which looks like this. And this distance here is $\frac{\phi}{\omega}$. OK?

So this just gives us a quick sketch of this function. And again, we note that we were able to plot this using the polar form very quickly. And the reason is because the polar form gives us the amplitude, the circular frequency, and the phase lag. Whereas if we were to take a look at the equivalent rectangular form, it's a little less obvious. The rectangular form is actually the sum of a sine and a cosine. And unless you can just add sines and cosines in your head very quickly, it's a little less obvious on how to come up with this picture.

OK. So just to summarize, we've used complex functions to analyze sinusoidal functions. And we've used polar form and rectangular form to get a handle on the complex functions. And specifically, complex functions are a very good way to represent sinusoidal functions. And just to reiterate, the reason is that they contain a lot of information that we can graphically handle and turn into a sketch. So I'll just conclude there, and I'll see you next time.