18.03SC Practice Problems 10

Homogeneous second order linear equations with constant coefficients

The exponential e^{rt} is a solution of $m\ddot{x} + b\dot{x} + kx = 0$ (where m, b, and k are real constants, and $m \neq 0$) exactly when r is a root of the *characteristic polynomial* $p(s) = ms^2 + bs + k$.

(1) *Overdamped*: Roots real and distinct: the general solution is given by linear combinations of these two exponentials.

(2) *Underdamped*: Roots not real: they are $-\dfrac{b}{2m} \pm \omega_d i$ where $\omega_d = \sqrt{\dfrac{k}{m} - \left(\dfrac{b}{2m}\right)^2}$. The corresponding exponential solutions are complex conjugates of each other. To get basic real solutions take the real and imaginary parts of either one (again solutions since $\operatorname{Re} z$ and $\operatorname{Im} z$ are linear combinations of z and $\overline{z}$). The result is $x_1 = e^{-bt/2m}\cos(\omega_d t)$ and $x_2 = e^{-bt/2m}\sin(\omega_d t)$. ω_d is called the *damped circular frequency*. The general real solution is thus given by real linear combinations of these two, or, what is the same, $e^{-bt/2m}$ times the general sinusoid of circular frequency ω_d: $Ae^{-bt/2m}\cos(\omega_d t - \phi)$.

(3) *Critically damped*: Roots equal (and hence real, $r = -b/2m$). Then there are not enough exponential solutions and the general solution is $(a + ct)e^{-bt/2m}$.

1. Start with $\ddot{x} + \omega^2 x = 0$. What is the characteristic polynomial? What are its roots? What are the exponential solutions – the solutions of the form $e^{\alpha t}$? These may be complex exponentials. What are their real and imaginary parts? Check that these are also solutions to the original equation. What is the general real solution?

2. Suppose that $e^{-t/2}\cos(3t)$ is a solution of the equation $m\ddot{x} + b\dot{x} + kx = 0$ (where m, b, k are real).

(a) What can you say about m, b, k?

(b) What are the exponential solutions (solutions of the form $e^{\alpha t}$) of this differential equation?

(c) Sketch the curve in the complex plane traced by one of the exponential solutions. Then sketch the graph of the real part, and explain the relationship.

(d) What is the general solution?

3. Let $\omega > 0$. A damped sinusoid $x(t) = Ae^{-at}\cos(\omega t)$ has "pseudo-period" $2\pi/\omega$. The pseudo-period, and hence ω, can be measured from the graph: it is twice the distance between successive zeros of $x(t)$, which is always the same. Now what is the spacing between successive maxima of $x(t)$? Is it always the same, or does it differ from one successive pair of maxima to the next?

4. Suppose that successive maxima of $x(t) = Ae^{-at}\cos(\omega t)$ occur at $t = t_0$ and $t = t_1$. What is the ratio $x(t_1)/x(t_0)$? (Hint: Compare $\cos(\omega t_0)$ and $\cos(\omega t_1)$.) Does this offer a means of determining the value of a from the graph?

5. For what value of b does $\ddot{x} + b\dot{x} + x = 0$ exhibit critical damping? For this value of b, what is the solution x_1 with $x_1(0) = 1$, $\dot{x}_1(0) = 0$? What is the solution x_2

with $x_2(0) = 0$, $\dot{x}_2(0) = 1$? (This is a "normalized pair" of solutions.) What is the solution such that $x(0) = 2$ and $\dot{x}(0) = 3$?