18.03SC Practice Problems 10

Homogeneous second order linear equations with constant coefficients

Solution suggestions

1. *Start with $\ddot{x} + \omega^2 x = 0$. What is the characteristic polynomial? What are its roots? What are the exponential solutions – the solutions of the form $e^{\alpha t}$? These may be complex exponentials. What are their real and imaginary parts? Check that these are also solutions to the original equation. What is the general real solution?*

The characteristic polynomial is $p(s) = s^2 + \omega^2$. Its roots are $\pm i\omega$. The exponential solutions are $e^{i\omega t}$ and $e^{-i\omega t}$. These are complex-valued functions, which are complex conjugates of each other. Up to sign, they have the same real and imaginary parts. Namely, $e^{\pm i\omega t} = \cos(\omega t) \pm i \sin(\omega t)$, so,

$$\mathrm{Re}\,(e^{i\omega t}) = \cos(\omega t), \quad \mathrm{Im}(e^{i\omega t}) = \sin(\omega t).$$

$$\mathrm{Re}\,(e^{-i\omega t}) = \cos(\omega t), \quad \mathrm{Im}(e^{-i\omega t}) = -\sin(\omega t).$$

We can check that these functions are also solutions by verifying that they satisfy the original equation:

$$\frac{d^2}{dt^2}(\cos(\omega t)) + \omega^2 \cos(\omega t) = -\omega^2 \cos(\omega t) + \omega^2 \cos(\omega t) = 0$$

and

$$\frac{d^2}{dt^2}(\sin(\omega t)) + \omega^2 \sin(\omega t) = -\omega^2 \sin(\omega t) + \omega^2 \sin(\omega t) = 0.$$

The same equation holds for $-\sin(\omega t)$, and, in fact for any linear combination of these two functions. The general real solution is

$$x(t) = a\cos(\omega t) + b\sin(\omega t).$$

In standard notation, this is the general sinusoid $A\cos(\omega t - \phi)$.

2. *Suppose that $e^{-t/2}\cos(3t)$ is a solution of the equation $m\ddot{x} + b\dot{x} + kx = 0$.*

(a) *What can you say about m, b, k?*

We can write $e^{-t/2}\cos(3t) = \mathrm{Re}\, e^{(-1/2 \pm 3i)t}$, so $p(s) = ms^2 + bs + k$ has solutions $-\frac{1}{2} \pm 3i$. This means $p(s) = m(s + 1/2 - 3i)(s + 1/2 + 3i) = m(s^2 + s + \frac{37}{4})$. Then m can be any nonzero number, $b = m$, and $k = \frac{37}{4}m$.

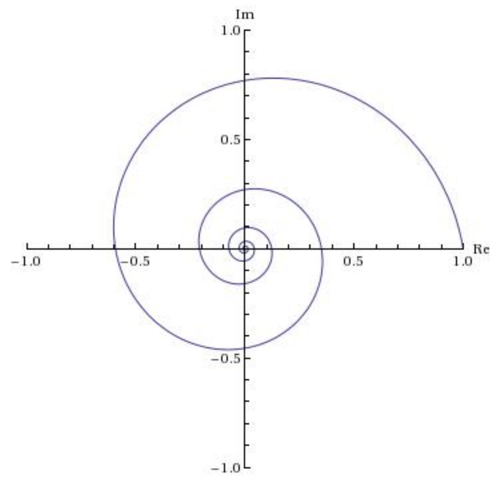
We could have also solved this problem by the technique we had always used before – by substitution into the equation, but then the calculations would have been harder. In general, you should always interpret knowing some exponential solutions of a homogeneous differential equation as knowing some of the roots of the characteristic polynomial.

(b) *What are the exponential solutions (solutions of the form $e^{\alpha t}$) of this differential equation?*

The exponential solutions are $e^{(-1/2+3i)t}$ and $e^{(-1/2-3i)t}$.

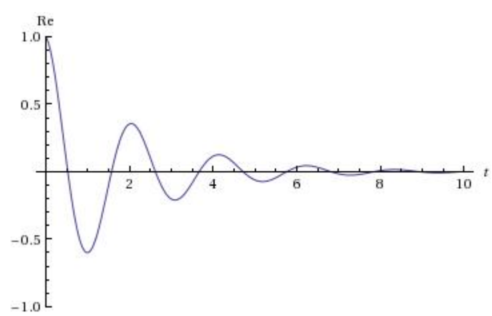
(c) *Sketch the curve in the complex plane traced by one of the exponential solutions. Then sketch the graph of the real part, and explain the relationship.*

Here is the trajectory traced by $e^{(-1/2+3i)t}$ in the complex plane.



The curve is a spiral. As t increases, it moves counterclockwise around the origin, and approaches zero.

Here is the graph of the real part as a function of t.



The graph of the real part describes a damped oscillation, i.e., it is roughly sinusoidal, with exponentially decreasing amplitude. It can be obtained from the graph of the complex-valued trajectory by envisioning moving along the trajectory in the complex plane as time t increases and projecting onto the real axis.

(d) *What is the general solution?*

All real-valued solutions are given by real linear combinations of the real part and the imaginary part of $e^{(-1/2+3i)t}$. So the general solution is $C_1 e^{-t/2}\cos(3t) +$

$C_2 e^{-t/2}\sin(3t)$ for any real numbers C_1 and C_2, or equivalently, $Ae^{-t/2}\cos(3t-\phi)$, where $A \geq 0$ and $\phi \in [0, 2\pi)$.

3. *Let $\omega > 0$. A damped sinusoid $x(t) = Ae^{-at}\cos(\omega t)$ has "pseudo-period" $2\pi/\omega$. The pseudo-period, and hence ω, can be measured from the graph: it is twice the distance between successive zeros of $x(t)$, which is always the same. Now what is the spacing between successive maxima of $x(t)$? Is it always the same, or does it differ from one successive pair of maxima to the next?*

The extrema of $x(t) = Ae^{-at}\cos(\omega t)$ occur when $\dot{x}(t) = 0$, i.e., $-a\cos(\omega t) = \omega\sin(\omega t)$. The extrema are achieved at t where $\tan(\omega t) = -a/\omega$. Since minima and maxima of $x(t)$ are alternating, the maxima occur at every other such t, and the spacing between successive maxima is twice the period of $\tan(\omega t)$, which is just the pseudo-period $2\pi/\omega$. This is always the same since it doesn't depend on t or x.

4. *Suppose that successive maxima of $x(t) = Ae^{-at}\cos(\omega t)$ occur at $t = t_0$ and $t = t_1$. What is the ratio $x(t_1)/x(t_0)$? (Hint: Compare $\cos(\omega t_0)$ and $\cos(\omega t_1)$.) Does this offer a means of determining the value of a from the graph?*

As seen in Question 3, $t_1 - t_0 = 2\pi/\omega$. So $\cos(\omega t_1) = \cos(\omega t_0 + 2\pi) = \cos(\omega t_0)$ are the same, and the ratio is given by $x(t_1)/x(t_0) = e^{-a(t_1-t_0)}$. So a can be estimated graphically using the formula $a = \frac{\ln x(t_0) - \ln x(t_1)}{t_1 - t_0}$, where $(t_0, x(t_0))$ and $(t_1, x(t_1))$ are any two successive maxima.

5. *For what value of b does $\ddot{x} + b\dot{x} + x = 0$ exhibit critical damping? For this value of b, what is the solution x_1 with $x_1(0) = 1$, $\dot{x}_1(0) = 0$? What is the solution x_2 with $x_2(0) = 0$, $\dot{x}_2(0) = 1$? (This is a "normalized pair" of solutions.) What is the solution such that $x(0) = 2$ and $\dot{x}(0) = 3$?*

The characteristic polynomial is $p(s) = s^2 + bs + 1$. For the system to be critically damped, the characteristic polynomial must be a perfect square, i.e., $p(s)$ must equal $(s-k)^2$ for some k. Multiplying and comparing gives $b = -2k$ and $k^2 = 1$. Therefore, $b = \pm 2$. When $b = -2$, e^t is a solution, and it exhibits exponential growth instead of damping, so we reject that value of b. So the system is critically damped when $b = 2$.

For this value of b, the general solution is $x(t) = (C_0 + C_1 t)e^{-t}$. The corresponding initial conditions for each solution are $x(0) = C_0$ and $\dot{x}(0) = -C_0 + C_1$. By using the given initial conditions to solve for the constants, we see that the normalized pair of solutions is $x_1 = e^{-t} + te^{-t}$ and $x_2 = te^{-t}$.

We can use these normalized solutions to read off the solution that satisfies any given inital condition. In particular, the solution with $x(0) = 2$ and $\dot{x}(0) = 3$ is

$$x = x(0)\cdot x_1 + \dot{x}(0)\cdot x_2 = 2x_1 + 3x_2 = 2e^{-t} + 5te^{-t}.$$

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