Part II Problems and Solutions

Problem 1: *[Superposition; exponential response formula] Find the general real solution of:*

(a) $\dot{x} + 2x = e^{3t}\cos(4t)$. *(You could do this using an integrating factor, but it's easier to write the equation as the real part of a complex-valued equation with exponential right hand side, and use the ERF.)*

(b) $\ddot{x} + \dot{x} + 2x = \cos t$. *(Express the sinusoidal solution of this equation in two ways: as* $a\cos t + b\sin t$*, and as* $A\cos(t-\phi)$*.)*

Now open the Mathlet `Forced Damped Vibrations`*. As in* `Damped Vibrations`*, the initial conditions are set by the box at left. There is now a forcing term,* $A\cos(\omega t)$*, and the values of* A *and* ω *are adjustable by sliders, as are the values of the mass, damping constant, and spring constant. The force in this system is applied directly to the mass (rather than being mediated by the spring or the dashpot).*

Set $m = 1.00$*,* $b = 1.00$*, and* $k = 2.00$*.*

(c) *Start by setting* $A = 0.00$*, so you are looking at the homogeneous case. Set the initial condition to* $x(0) = 0$ *and* $\dot{x}(0) = 1$*, select the Solution curve only, and measure the pseudoperiod. (Notice that a slider above the graphing window lets you adjust the vertical scale.) How well does your measurement agree with what you found in* **(b)***?*

(d) *Now set* $A = 1.00$ *and* $\omega = 1.00$*. Select the green Steady State curve. Measure its amplitude and the value at* $t = 0$*. You computed this solution in* **(b)***. Compare the computed amplitude and value at* $t = 0$ *with what you measured. Also compute* $\dot{x}_p(0)$*.*

(e) *Finally, select initial condition* $x(0) = 0$*,* $\dot{x}(0) = 0$*: so-called* rest initial conditions*. (Clicking on a hashmark gives the exact value.) If you select Steady State, Transient, and Solution, you can see all three:* $x = x_p + x_h$*. Please compute* x_h*.*

Solution: **(a)** $\dot{z} + 2z = e^{(3+4i)t}$ has solution

$$z_p = \frac{e^{(3+4i)t}}{p(3+4i)} = \frac{e^{(3+4i)t}}{5+4i} = \frac{1}{\sqrt{41}}e^{3t}e^{i(4t-\phi)},$$

where $\phi = \mathrm{Arg}(5+4i) = \tan^{-1}(4/5)$. Thus,

$$x_p = \mathrm{Re}\,(z_p) = \frac{e^{3t}}{\sqrt{41}}\cos(4t-\phi).$$

The homogeneous equation has solution $x_h = ce^{-2t}$, so

$$x = \frac{e^{3t}}{\sqrt{41}}\cos(4t-\phi) + ce^{-2t}.$$

(b) $\ddot{z}+\dot{z}+2z=e^{it}$ has solution $z_p=e^{it}/p(i)$; $p(i)=-1+i+2=1+i$ so $z_p=\frac{1}{1+i}e^{it}=\frac{1-i}{2}(\cos t+i\sin t)$ and $x_p=\frac{1}{2}\cos t+\frac{1}{2}\sin t$. We can also write $p(i)=1+i=\sqrt{2}e^{i\pi/4}$, so $z_p=\frac{1}{\sqrt{2}}e^{-i\pi/4}e^{it}=\frac{\sqrt{2}}{2}e^{i(t-\pi/4)}$ and $x_p=\mathrm{Re}\,(z_p)=\frac{\sqrt{2}}{2}\cos(t-\frac{\pi}{4})$. The characteristic polynomial is $p(s)=s^2+s+2=(s+\frac{1}{2})^2+\frac{7}{4}$ with roots $-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$, so the homogeneous equation has general solution $x_h=e^{-t/2}\left(a\cos\left(\frac{\sqrt{7}}{2}t\right)+b\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$ or $x_h=Ae^{-t/2}\cos\left(\frac{\sqrt{7}}{2}t-\phi\right)$. The general solution is $x=x_p+x_h$.

(c) I measure the first six zeros to be $2.36, 4.76, 7.12, 9.48, 11.89, 14.19$. The successive differences are $2.40, 2.36, 2.36, 2.41, 2.30$, with an average of 2.366. My measured pseudoperiod is twice this, 4.73. The damped circular frequency of this system is $\omega_d=\frac{\sqrt{7}}{2}$ so the pseudoperiod is $\frac{2\pi}{\omega_d}=\frac{4\pi}{\sqrt{7}}\simeq 4.7496$. Not bad agreement.

(d) I measure the amplitude as 0.71. It looks like $x_p(0)\simeq 0.50$. Computed amplitude is $\frac{\sqrt{2}}{2}\simeq 0.707$. Computed value is $x_p(0)=\frac{\sqrt{2}}{2}\cos(-\frac{\pi}{4})=\frac{1}{2}$. Good agreement! $\dot{x}_p(t)=-\frac{\sqrt{2}}{2}\sin(t-\frac{\pi}{4})$ so $\dot{x}_p(0)=-\frac{\sqrt{2}}{2}\sin(-\frac{\pi}{4})=\frac{1}{2}$.

(e) We need to select x_h so that $x_h(0)=-x_p(0)=-\frac{1}{2}$ and $\dot{x}_h(0)=-\dot{x}_p(0)=-\frac{1}{2}$. It's more convenient to use the rectangular expression for x_h for this. $x_h(0)=a$ so $a=-\frac{1}{2}$. $\dot{x}_h=e^{-t/2}\left(\left(-\frac{1}{2}a+\frac{\sqrt{7}}{2}b\right)\cos\left(\frac{\sqrt{7}}{2}t\right)+(\cdots)\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$ so $\dot{x}_h(0)=-\frac{1}{2}a+\frac{\sqrt{7}}{2}b$. Thus $b=\frac{2}{\sqrt{7}}\left(\dot{x}_h(0)+\frac{1}{2}a\right)=\frac{2}{\sqrt{7}}\left(-\frac{1}{2}-\frac{1}{4}\right)=-\frac{3}{2\sqrt{7}}$. $x_h=e^{-t/2}\left(-\frac{1}{2}\cos\left(\frac{\sqrt{7}}{2}t\right)-\frac{3}{2\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$.

Problem 2: *[Operators, resonance] Use the Resonant ERF to find the general real solution of*

$$\dot{x}+x=e^{-t}$$

Solution: $p(s)=s+1$ and $p(-1)=0$, so we are in resonance. $p'(s)=1$ so the ERF/Resonant gives

$$x_p=te^{-t}.$$

The general solution is therefore

$$x=te^{-t}+ce^{-t}.$$

18.03SC Differential Equations
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