Generalized Exponential Response Formula

We can also solve an LTI DE $p(D)x = q(t)$ with exponential input $q(t) = Be^{at}$ even when $p(a) = 0$. The answer is given by the following **generalized Exponential Response** formula (the proof of which we postpone to the session on Linear Operators.

Generalized Exponential Response Formula. *Let $p(D)$ be a polynomial operator with constant coefficients, and $p^{(s)}$ its s-th derivative. Then*

$$p(D)x = Be^{at}, \qquad \text{where } a \text{ is real or complex}$$

has the particular solution

$$x_p = \begin{cases} \dfrac{Be^{at}}{p(a)} & \text{if } p(a) \neq 0 \\ \dfrac{Bte^{at}}{p'(a)} & \text{if } p(a) = 0 \text{ and } p'(a) \neq 0 \\ \dfrac{Bt^2e^{at}}{p''(a)} & \text{if } p(a) = p'(a) = 0 \text{ and } p''(a) \neq 0 \\ \dots & \\ \dfrac{Bt^se^{at}}{p^{(s)}(a)} & \text{if } a \text{ is an } s\text{-fold zero} \end{cases}$$

Note: Later when we cover resonance the case $p(a) = 0$, $p'(a) \neq 0$ will be called the *Resonant Response Formula*

Example 1. Find a particular solution to the equation

$$\ddot{x} + 8\dot{x} + 15x = e^{-5t}$$

Solution. The characteristic polynomial is $p(r) = r^2 + 8r + 15$. Since $p(-5) = 0$ we need to use the generalized ERF.
Computing $p'(r) = 2r + 8$, which implies $p'(-5) = -2$. Therefore the generalized ERF gives

$$x_p = \frac{te^{-5t}}{p'(-5)} = -\frac{te^{-5t}}{2}.$$

Example 2. Find a particular solution to

$$\ddot{x} + 2\dot{x} + 2x = e^{-t}\cos t.$$

Solution. First we complexify the equation

$$\ddot{z} + 2\dot{z} + 2z = e^{(-1+i)t}, \qquad \text{where} \qquad x = \text{Re}(z).$$

The characteristic polynomial is $p(r) = r^2 + 2r + 2$. Computing,

$$p(-1+i) = (-1+i)^2 + 2(-1+i) + 2 = 0, \quad p'(r) = 2r + 2, \quad p'(-1+i) = 2i.$$

Since $p(-1+i) = 0$ we use the generalized ERF

$$z_p = \frac{te^{(-1+i)t}}{p'(-1+i)} = \frac{te^{(-1+i)t}}{2i} = \frac{te^{-t}(\cos t + i \sin t)}{2i}$$

Finally we take the real part to get

$$x_p = \text{Re}(z_p) = \frac{te^{-t} \sin t}{2}.$$