Part I Problems and Solutions

Find the general solution to the given DE and also the specific solution satisfying the given initial conditions (if any).

Problem 1: $y'' - 3y' + 2y = 0$

Solution: Characteristic equation $p(r) = r^2 - 3r + 2 = 0$ or $(r-1)(r-2) = 0$ with roots $r = 1, 2$. Thus we have the general solution

$$y = c_1 e^x + c_2 e^{2x}$$

Problem 2: $y'' + 2y' - 3y = 0, \qquad y(0) = 1, \; y'(0) = -1.$

Solution: Characteristic equation $r^2 + 2r - 3 = 0$, or $(r+3)(r-1) = 0$, so $y = c_1 e^x + c_2 e^{-3x}$. Put in initial conditions:

$$y(0) = 1 \Rightarrow c_1 + c_2 = 1$$
$$y'(0) = -1 \Rightarrow c_1 - 3c_2 = -1$$

Solve for c_1, c_2, and we get

$$y = \frac{1}{2}e^x + \frac{1}{2}e^{-3x}$$

In the next three problems, find a DE of the form $ay'' + by' + cy = 0$ which has the given family of solutions $y = c_1 y_1 + c_2 y_2$, with c_1, c_2 constant.

Problem 3: $y = c_1 + c_2 e^{-5x}$

Solution: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ with r_1, r_2 roots of $p(r) = ar^2 + br + c \to P(r) = (r - r_1)(r - r_2)$.

In this problem, $r_1 = 0, r_2 = -5$, so $p(r) = (r-0)(r+5) = r^2 + 5r$. Thus, $a = 1, b = 5, c = 0$, so the DE $y'' + 5y' = 0$ has these solutions.

Problem 4: $y = c_1 e^{5x} + c_2 e^{-5x}$

Solution: $r_1 = 5, r_2 = -5$ so $p(r) = (r-5)(r+5) = r^2 - 25 = ar^2 + br + c$ so $a = 1, b = 0, c = -25$, and so the DE $y'' - 25y = 0$ has these solutions.

Problem 5: $y = c_1 + c_2 x$

Solution: $y' = c_2$ and $y'' = 0$, so we can take $a = 1, b = c = 0$. Thus the DE $y'' = 0$ has these solutions.

In the next four problems, find the general solution of the given DE.

Problem 6: $y'' - 4y = 0$

Solution: Characteristic equation $p(r) = r^2 - 4 = 0$ so roots are ± 2, and so $y = c_1 e^{2x} + c_2 e^{-2x}$ is the general solution.

Problem 7: $2y'' - 3y' = 0$

Solution: Characteristic equation $p(r) = 2r^2 - 3r = 0$ has roots $0, \frac{3}{2}$, and so the general solution is

$$y = c_1 + c_2 e^{\frac{3}{2}x}$$

Problem 8: $4y'' - 12y' + 9y = 0$

Solution: Characteristic equation $p(r) = 4r^2 - 12r + 9 = (2r - 3)^2 = 0$ so we have one repeated root $r = \frac{3}{2}$. Thus the solutions are

$$y = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}$$

Problem 9: $y^{(4)} - 8y'' + 16y = 0$

Solution: Characteristic equation $p(r) = r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r - 2)^2(r + 2)^2 = 0$. This has double roots at $r = \pm 2$, so the solutions are:

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}$$

Find the general solution to the general DE and also the one satisfying the initial conditions (if any are given).

Problem 10: $y'' + 2y' + 2y = 0$

Solution: Char. eqn. $r^2 + 2r + 2 = 0$

By quadratic formula, $r = -1 \pm i$, so the general solution is
$y = e^{-x}(c_1 \cos x + c_2 \sin x)$

(using as y_1, y_2 the real and imaginary parts of the characteristic solution $y = e^{(-1+i)x} = e^{-x}(\cos x + i \sin x)$)

Problem 11: $y'' - 2y' + 5y = 0; \qquad y(0) = 1,\ y'(0) = -1$

Solution: Characteristic equation $r^2 - 2r + 5 = 0$. By quadratic formula, $r = 1 \pm 2i$.

General solution is thus $y = e^x(c_1 \cos 2x + c_2 \sin 2x)$

Putting in initial conditions (you'll need to find y' first!):

$$\begin{aligned} y(0) = 1 &\Rightarrow c_1 = 1 \\ y'(0) = 1 &\Rightarrow c_1 + 2c_2 = -1 \rightarrow c_2 = -1 \end{aligned}$$

so

$$y = e^x (\cos 2x - \sin 2x)$$

Problem 12: $y'' - 4y' + 4y = 0; \qquad y(0) = 1,\ y'(0) = 1$

Solution: Characteristic equation $r^2 - 4r + 4 = 0$ or
$(r-2)^2 = 0$; $r = 2$ double root. So $y = e^{2x}(c_1 x + c_2)$ is the general solution. Put in initial conditions:

$$\begin{aligned} y(0) = 1 &\Rightarrow c_2 = 1 \\ y'(0) = 1 &\Rightarrow 2c_2 + c_1 = 1 \rightarrow c_1 = -1 \end{aligned}$$

So the solution is $\quad y = (1 - x)e^{2x}$.

Problem 13: *Find the general solution to the DE* $\ y'' + 6y' + 9y = 0$

Solution: Characteristic equation $r^2 + 6r + 9 =$
$(r+3)^2 = 0$ has a double root $r = -3$ so the general solution is $\ y = c_1 e^{-3x} + c_2 x e^{-3x}$.

In the next two problems, solve the given initial-value problem.

Problem 14: $y'' - 4y' + 3y = 0, \qquad y(0) = 7,\ y'(0) = 11.$

Solution: Characteristic equation $r^2 - 4r + 3 = 0 \to$
$(r-3)(r-1) = 0$ with roots $r = 1, 3$ so the general solution to this DE is

$$y = c_1 e^x + c_2 e^{3x}$$

IC's:

$$\left.\begin{array}{l} y(0) = c_1 + c_2 = 7 \\ y'(0) = c_1 + 3c_2 = 11 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 5 \\ c_2 = 2 \end{array} \Rightarrow y = 5e^x + 2e^{3x}$$

Problem 15: *$y'' - 6y' + 25y = 0, \qquad y(0) = 3, \; y'(0) = 1$*

Solution: Characteristic equation $r^2 - 6r + 25 = 0$ has roots $r = \frac{1}{2}\left(6 \pm \sqrt{36 - 100}\right) = \frac{1}{2}(6 \pm 8i) = 3 \pm 4i$.

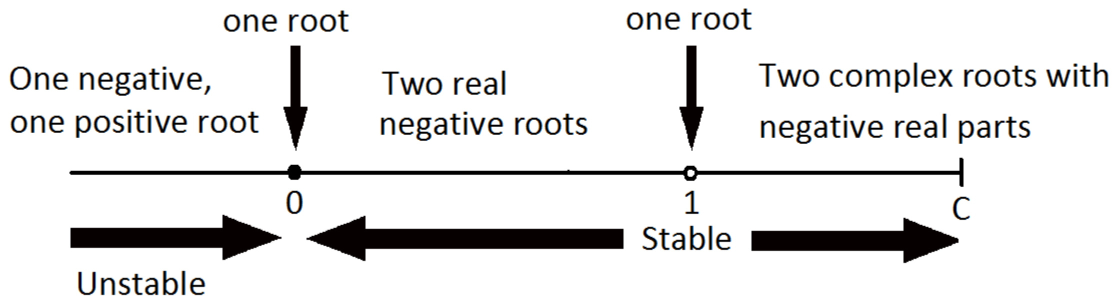
Real solutions are $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$.

IC's: $y(0) = c_1 = 3$, $y'(0) = 3c_1 + 4c_2 = 1$ gives $c_1 = 3, c_2 = -2$, so the solution is

$$y = e^{3x}\left(3\cos 4x - 2\sin 4x\right)$$

Problem 16: *For the equation $y'' + 2y' + cy = 0$, $\quad c$ constant,*
a) Tell which values of c correspond to each of the three cases: two real roots, repeated real root, and complex roots.
b) For the case of two real roots, tell for which values of c both roots are negative, both roots are positive, or the roots have different signs.
c) Summarize the above information by drawing a c-axis, and marking the intervals on it corresponding to the different possibilities for the roots of the characteristic equation.
d) Finally, use this information to mark the interval on the c-axis for which the corresponding ODE is stable. (The stability criterion using roots is what you will need.)

Solution: $y'' + 2y' + cy = 0$ has characteristic equation $r^2 + 2r + c = 0$, with roots $-1 \pm \sqrt{1-c}$. Below 0, there is one negative and one positive root. At 0, there is one root. Between 0 and 1, there are two real negative roots. At 1, there is one negative root. Greater than 1, there are two complex roots with negative real part. The region below 0 is unstable; the rest of the axis is stable.


one root
one root
One negative,
one positive root
Two real
negative roots
Two complex roots with
negative real parts
0
1
C
Stable
Unstable