

PROFESSOR: Welcome back. So in this session we're going to cover homogeneous constant coefficient equations for any roots. So we're told to assume that z of t is equal to exponential minus $3t$ times cosine plus i of sine t . And to assume that this complex number is a solution to this differential equation, which is second order with constant coefficients m , b , and k , which are real, and from this assumption give two real solutions to this equation.

In the second part, we're asked to find a general solution for this other differential equation, which is the same form as that seen in part a, with now the real values for b equals to 6 , m equals to 1 , and k equals to 10 . And then we're asked of this system that would be captured by this differential equation is overdamped or underdamped.

In the last part we switch gears and we're given a series of roots, eight roots to an eighth degree polynomial, and we're asked to write down the general solution for that polynomial. Note that here we have repeated roots, which is basically where the trick is. So why don't you take a few minutes, and we'll come back to solve these problems.

Welcome back. So for the first part, we're given a complex number clearly split into its real part and its imaginary part. So this is the imaginary part of z , and this is the real part of z . Now, the differential equation that we are given is constant value-- differential equation second order, linear, and homogeneous. So there is no right-hand side. So clearly you can see that if a complex number that can be written as real part plus i imaginary part is a solution to this equation, then both sides independently are also solutions to this equation.

So by giving us this complex number, we are actually given two types of solutions that we can then write down as two linearly independent solutions. So the general solution to this differential equation can then be written in the form of-- one constant coefficient is undetermined, minus $3t \cos t$ plus another constant coefficient minus $3t \sin t$. Where basically, we're introducing a linear combination of two linearly independent solutions that were just taken from the original complex number we were given.

So for the second part now, we are asked to find a general solution for the ODE. We'll just rewrite here for now. So again, constant coefficient, homogeneous equation. And we're just asked to find a general solution. So we learned from the class notes that we can use the characteristic polynomial method to solve this equation.

And the characteristic polynomial here will have this form. And basically, we just need to find the roots of this characteristic polynomial to be able then to express the solutions of this homogeneous constant coefficient equation in terms of function, exponential to whatever roots we will find.

So I just want to write down the discriminant of this polynomial. So what do we have here? What was the question again? So we're asked then to just find the discriminant for this equation. So $b^2 - 40$, which is basically -4 . So we have a discriminant that is negative. And the roots for our polynomial here are just going to be... the root absolute value of the discriminant with the complex number i out, because we had a negative discriminant.

So these couple of solutions, plus or minus, tell us that we are going to have oscillations as solutions for this differential equation, and that it's also going to be actually damped oscillations, because we have the first-- the real part of the number being negative.

So let me just write down the pair of solutions that we would have. We will have this that comes out of the real part, cosine. I'll put the root of 4 is just 2. And so we have $\cos t$, that's one. $e^{-t/2}$, of course, is just $e^{-t/2}$.

So here I switched directly to writing them in cosine and sine, in the forms. But we could also have kept it in the complex form if we were solving the equation in complex space. And then we would just have complex exponentials here with the frequencies ω -- angular frequency 1.

So this is the homogeneous equation, the homogeneous solution for this homogeneous constant coefficient differential equation. Now, what do we see? We can see that it's exactly the same form that the solution that we had previously.

And so the question here is this oscillator overdamped or underdamped? And as we see here, we're going to have oscillations that are damped by a pre-factor of decaying exponential e^{-3t} for both cases. So this is decaying. So there is definitely damping. And we can see that here, because we had a positive value for b , which is the damping term in the oscillator system.

But there are still oscillations, because we had a discriminant that was negative, which introduced complex numbers, hence cosines and sines, so basically, oscillations. Therefore we're in the case of an underdamped response, an underdamped system. The discriminant

was negative.

So for the last part of this problem, we now leave this differential equation, which was second order, and move to higher-order differential equations of eighth order. And we're given eight roots. $4 \pm 5i$.

We have the root of 3, which is real, which is repeated three times, and a root $4 \pm 5i$, which is repeated two times. So we saw before then when we have repeated roots, that means that we would have, let's say, in a second-order differential equation. We would have in this case a discriminant that would be equal to 0. And we would need to introduce a different type of solution, because instead of introducing an exponential of e^{rt} , exponential minus e^{-rt} , now we don't have the minus e^{-rt} . So we need to seek another type of solution. And we'll multiply the exponential by t .

So one type of solution for this unique root that would come out would be just $2t$. For the first of the tripled root, it would be a $3t$. For the second one, we would have t^2 . For the third one, we need to go higher order, t^3 .

Now we have complex roots, so that gives us solutions of the same form that we found in questions a and b. So I'm going to just write them here, where now we have $e^{4t} \cos 5t$ plus $e^{4t} \sin 5t$. And for the repeated root, it's the same game. Even though we're in complex roots, it's the same idea, we need to multiply all of it by t .

So these are the families of linearly independent solutions that would come out of these roots. So now the general solution is simply the linear combination of all these linearly independent functions. So we will introduce different constants. c_5, c_6, c_7 , and c_8 . And we have eight undetermined coefficients to deal with, because these would be eight roots of an eighth-order polynomial. And so if we were to solve an initial value problems like that, we would need eight different types of initial conditions to determine the values of these constants.

So to summarize, we used the characteristic polynomial method to solve these problems introducing higher-order equations than the first-order differential equations we saw before. And what is important to remember is that the different types of solutions that you get, depending on the roots of the characteristic polynomial and the sign of the discriminant. So whether the roots are real or purely imaginary or imaginary numbers will give three types of the behavior: the critically damped behavior; if it's real then it's going to be just overdamped behavior; and if the roots are complex, then it would be underdamped behavior.

Now when we are given different types of roots, with repeated roots it's important to know how to construct the solutions as well. And that's by introducing new types of functions where we're multiplying by polynomial in t in front of the exponential. So that ends the session for today.