18.03SC Practice Problems 21

Fourier Series: Introduction

Solution suggestions

This problem session is intended as preparation for working with Fourier series.

1. *What is the general solution to $\ddot{x} + \omega_n^2 x = 0$? Try to remember it rather than deriving it again.*

This is an important system for us, and the goal is to get to the point of being able to recognize any homogeneous equation of this form and to remember that it has the general solution

$$x(t) = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t),$$

or, equivalently,

$$x(t) = A\cos(\omega_n t - \phi).$$

In our previous terminology, this is the second order linear homogeneous differential equation modeling a simple (undamped) harmonic oscillator with natural frequency ω_n. It has purely sinusoidal homogeneous response, with amplitude and phase shift that are determined by the inital conditions.

2. *Verify that (as long as $\omega \neq \pm\omega_n$)*

$$x_p = a\,\frac{\cos(\omega t)}{\omega_n^2 - \omega^2} \quad \textit{is a solution to} \quad \ddot{x} + \omega_n^2 x = a\cos(\omega t),$$

and that

$$y_p = b\,\frac{\sin(\omega t)}{\omega_n^2 - \omega^2} \quad \textit{is a solution to} \quad \ddot{y} + \omega_n^2 y = b\sin(\omega t).$$

The point here is that

$$\frac{d^2}{dt^2}\left(\cos(\omega t)\right) = \frac{d}{dt}\big(-\omega\sin(\omega t)\big) = -\omega^2\cos(\omega t)$$

and

$$\frac{d^2}{dt^2}\left(\sin(\omega t)\right) = -\omega^2\sin(\omega t).$$

We can verify that x_p satisfies the first equation by plugging it in:

$$\begin{aligned}
\ddot{x}_p + \omega_n^2 x_p &= \frac{d^2}{dt^2}\left(\frac{a}{\omega_n^2 - \omega^2}\cos(\omega t)\right) + \omega_n^2 \cdot \frac{a}{\omega_n^2 - \omega^2}\cos(\omega t) \\
&= \frac{a}{\omega_n^2 - \omega^2}\cdot(-\omega^2)\cdot\cos(\omega t) + \frac{a\omega_n^2}{\omega_n^2 - \omega^2}\cos(\omega t) = \frac{-a\omega^2 + a\omega_n}{\omega_n^2 - \omega^2}\cos(\omega t) \\
&= a\cos(\omega t).
\end{aligned}$$

Similarly, we can see that y_p satisfies the second equation:

$$\ddot{y_p} + \omega_n^2 y_p = \frac{b}{\omega_n^2 - \omega^2} \cdot (-\omega^2) \cdot \sin(\omega t) + \frac{b\omega_n^2}{\omega_n^2 - \omega^2} \sin(\omega t) = b \sin(\omega t).$$

That is, as in Problem 1, we are again relying on the simple, but important, observation that both sine and cosine are proportional to their second derivatives.

Remark what is happening: Here we are driving a simple harmonic oscillator by purely sinusoidal inputs, which themselves satisfy harmonic oscillator equations. If we do this at input frequencies ω away from the natural frequency ω_n, as we do here, the output will also be sinusoidal, with no phase lag, scaled by the gain of the system, $\frac{1}{\omega_n^2 - \omega^2}$.

The case of driving at the natural frequency (driving in resonance) is examined in the next problem.

3. *What about $\ddot{x} + \omega_n^2 x = \cos(\omega_n t)$? What is a particular solution? What is the general solution? Are there any solutions $x(t)$ such that $|x(t)| < 10^6$ for all t? Are there any periodic solutions?*

When $\omega = \omega_n$, we cannot use the solutions suggested in Problem 2. So we proceed as usual and solve from the beginning.

Take a complex replacement and solve the complex-valued equation $\ddot{z} + \omega_n^2 z = e^{i\omega_n t}$. Here $p(i\omega_n) = 0$, so the standard ERF does not apply. The first derivative $p'(i\omega_n) = 2i\omega_n = 0$, so the (first-order) Resonant ERF does apply, giving us a particular solution to the complex equation,

$$z_p(t) = \frac{te^{i\omega_n t}}{2i\omega_n}.$$

Therefore, $x_p(t) = \mathrm{Re}(\mathrm{z_p(t)}) = \frac{t\sin(\omega_n t)}{2\omega_n}$ is a particular solution to the original equation.

Combining this with the homogeneous solution we found in Problem 1, we get that the general solution is

$$x(t) = \frac{t\sin(\omega_n t)}{2\omega_n} + c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t).$$

The next part of this question is asking about the existence of bounded solutions. Every homogeneous component here is bounded, while the particular solution we found, x_p, has a factor of t in its numerator (is oscillating-linear). So any solution $x(t)$ oscillates through unboundedly increasing values of x as t increases, and will eventually leave any bounding range, including $(-10^6, 10^6)$, for some sufficiently large values of t. That is, there are no solutions $x(t)$ such that $|x(t)| < 10^6$ for all t.

The last part of this question asks whether there are any periodic solutions. Every homogeneous component here is a periodic function with period $2\pi/\omega_n$, but the particular solution we found, $x_p(t) = \frac{t\sin(\omega_n t)}{2\omega_n}$, is not periodic. So the general solution $x(t) = x_p(t) + c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$ is not periodic for any c_1, c_2.

That is, a differential equation that models driving a harmonic oscillator in resonance has no periodic solution. Instead, all solutions oscillate with unboundedly increasing amplitude.

Note: We have been implicitly assuming that the natural frequency ω_n is nonzero. If ω_n were zero, our equation would have the form $\ddot{x} = 0$, which is again a situation of resonance. This is a separable equation. By integrating twice, we see that it has the general solution $x = c_1 t + c_2$. So here also there are no periodic (nonconstant) solutions, but in this case there are no oscillating solutions either.

4. *On the same set of axes, sketch the graphs of* $\sin t$ *and* $\sin(2t)$. *Then sketch the graph of* $f(t) = \sin t + \sin(2t)$.

Some pointers: $f(t)$ *is easy to evaluate when one of the terms is zero. What is the derivative at points where both terms are zero? This information should be enough to let you make a rough sketch.*

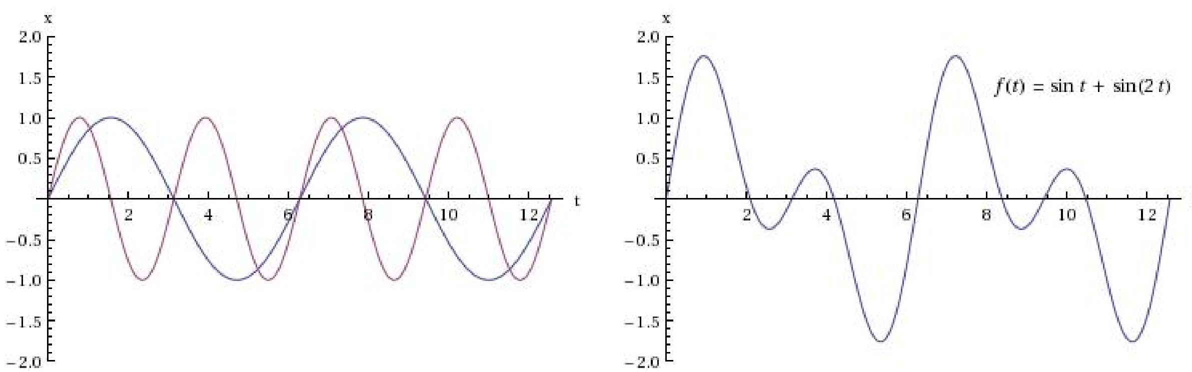
What are the periods of these three functions?

The function $\sin t$ has period 2π, and the function $\sin(2t)$ has period π.

Both $\sin t$ and $\sin(2t)$ vanish at $t = k\pi$ for $k \in \mathbb{Z}$. $f'(t) = \cos t + 2\cos(2t)$, so at those points, $f'(k\pi) = (-1)^k + 2 = \begin{cases} 1, & k\,odd, \\ 3, & k\,even. \end{cases}$

The function $f(t)$ is a linear combination of $\sin t$ and $\sin(2t)$, so its period is the least common multiple of the periods of $\sin t$ and $\sin(2t)$ – i.e., 2π.

Using this information, sketch the two functions and their sum. You can check your work against the computer-generated graphs in the figure below.



The graph of $\sin t$ and $\sin(2t)$ over a domain of two common periods, followed by a graph of their sum $f(t)$.

5. *For what values of* ω_n *is there a periodic solution to the equation*

$$\ddot{x} + \omega_n^2 x = b_1 \sin t + b_2 \sin(2t)$$

(where b_1 *and* b_2 *are nonzero)? Name one if it exists.*

Use linearity and our previous work to solve.

By linearity, a solution to this equation can be expressed as the sum of a solution to $\ddot{x} + \omega_n^2 x = b_1 \sin t$ and a solution to $\ddot{x} + \omega_n^2 x = b_2 \sin(2t)$.

If ω_n is not 1 or 2, we know from Problem 2 that each of these two equations have periodic particular solutions, and their sum

$$x_p = \frac{b_1 \sin t}{\omega_n^2 - 1} + \frac{b_2 \sin(2t)}{\omega_n^2 - 4}$$

is also periodic, of period 2π.

Now, if ω_n is either 1 or 2, one of the input frequencies is in resonance with the natural frequency of our system, and, by linearity, any solution to this differential equation will have a non-periodic summand – a sinusoid multiplied factor of t, as determined in Problem 3. In this case there are no periodic solutions.

Thus there is always a periodic solution unless $\omega_n = 1$ or $\omega_n = 2$.