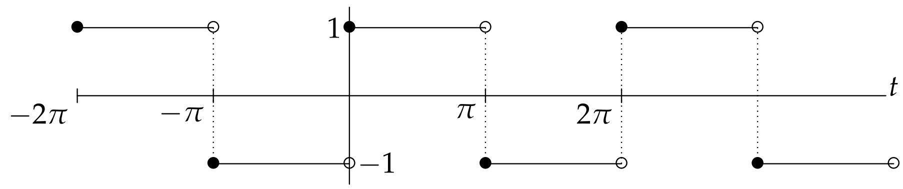
Examples

Example 1. Compute the Fourier series of $f(t)$, where $f(t)$ is the *square wave* with period 2π. which is defined over one period by

$$f(t) = \begin{cases} -1 & \text{for } -\pi \leq t < 0 \\ 1 & \text{for } 0 \leq t < \pi \end{cases} .$$

The graph over several periods is shown below.



Solution. Computing a Fourier series means computing its Fourier coefficients. We do this using the integral formulas for the coefficients given with Fourier's theorem in the previous note. For convenience we repeat the theorem here.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, \cos nt\, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, \sin nt\, dt$$

In applying these formulas to the given square wave function, we have to split the integrals into two pieces corresponding to where $f(t)$ is $+1$ and where it is -1. We find

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt = \int_{-\pi}^{0} (-1) \cdot \cos(nt)\, dt + \int_{0}^{\pi} (1) \cdot \cos(nt)\, dt.$$

Thus, for $n \neq 0$:

$$a_n = -\frac{\sin(nt)}{n\pi}\bigg|_{-\pi}^{0} + \frac{\sin(nt)}{n\pi}\bigg|_{0}^{\pi} = 0$$

and for $n = 0$:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt = 0.$$

(Note: it's advisable to do a_0 separately.)

Likewise

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)\,dt \\ &= \left.\frac{\cos(nt)}{n\pi}\right|_{-\pi}^{0} - \left.\frac{\cos(nt)}{n\pi}\right|_{0}^{\pi} = \frac{1-\cos(-n\pi)}{n\pi} - \frac{\cos(n\pi)-1}{n\pi} \\ &= \frac{2}{n\pi}(1-\cos(n\pi)) = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}. \end{aligned}$$

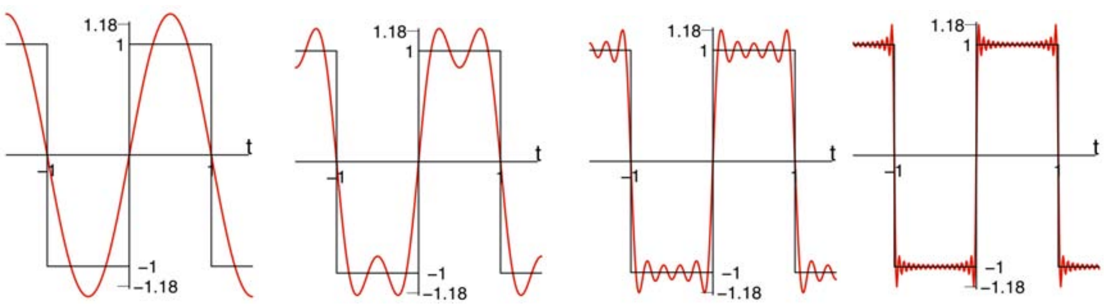
We have used the simplification $\cos n\pi = (-1)^n$ to get a nice formula for the coefficients b_n. (Note: when you get $\cos n\pi$ in these calculations it's *always useful* to make this substitution.)

This then gives the Fourier series for $f(t)$:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right).$$

Example 2. seeing the convergence of a Fourier series

The claim is that $f(t) = \dfrac{4}{\pi}\left(\sin t + \dfrac{1}{3}\sin 3t + \dfrac{1}{5}\sin 5t + \cdots\right)$. However, it is not easy to see that the sum on the right-hand side is in fact converging to the square wave $f(t)$. So let's use a computer to plot the sums of the first N terms of the series. for N = 1, 3, 9, 33. We get the following four graphs:



Notice that since a finite sum of sine functions is continuous (in fact smooth), the partial sums cannot jump when t is an integer multiple of π, the way the square way $f(t)$ does. But they are certainly "trying" to become the square wave $f(t)$! And the more terms you add in, the better the fit, with the theoretical limit as $N \to \infty$ being exactly equal to $f(t)$ (except actually at the jumps $t = n\pi$, as we'll see).

Note: In this case we don't have any cosine terms, just sine. This turns out to be not an accident: it follows from the fact that $f(t)$ here is an *odd* function, i.e. $f(-t) = -f(t)$, and such functions have only sines (which are also odd functions) in their Fourier series. Similarly for even functions and cosine series: if $f(t)$ is even ($f(-t) = f(t)$) then all the b_n's vanish and the Fourier series is simply $f(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt)$; while if $f(t)$ is odd then all the a_n's vanish and the Fourier series is $f(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$.