Part II Problems and Solutions

Problem 1: *[Periodic solutions] Let $g(t)$ be the function which is periodic of period 2π, and such that $g(t) = t$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$ and $g(t) = \pi - t$ for $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.*

(a) *Find a periodic solution to $\ddot{x} + \omega_0^2 x = g(t)$ (if there is one).*

(b) *For what (positive) values of ω_0 are there no periodic solution?*

(c) *Write ω_r for the smallest number you found in* **(b)**. *For ω_0 just less than ω_r, what is the solution like, approximately? How about for ω_0 just larger than ω_r?*

(d) *For what values of ω_0 are there more than one periodic solution?*

(e) *For the values of ω_0 found in* **(d)**, *are* all *solutions to $\ddot{x} + \omega_0^2 x = g(t)$ periodic?*

Solution: **(a)** Using the integral formulas for the coefficients as usual we first calculate the Fourier series for $g(t)$. The result is

$$g(t) = \frac{4}{\pi}\left(\sin(t) - \frac{1}{3^2}\sin(3t) + \frac{1}{5^2}\sin(5t) - \cdots\right).$$

By superposition and the fact that

$$\ddot{x} + \omega_0^2 x = A\sin(\omega t) \quad \text{has solution} \quad A\frac{\sin(\omega t)}{\omega_0^2 - \omega^2}$$

we find that a solution to $\ddot{x} + \omega_0^2 x = g(t)$ is given by

$$x_p = \frac{4}{\pi}\left(\frac{\sin(t)}{\omega_0^2 - 1^2} - \frac{1}{3^2}\frac{\sin(3t)}{\omega_0^2 - 3^2} + \cdots\right),$$

as long as ω_0 is not an odd integer.

(b) If ω_0 is an odd integer there is no periodic solution.

(c) $\omega_r = 1$. For ω just less than 1, the term $\frac{4}{\pi}\frac{\sin(t)}{\omega_0^2 - 1}$ dominates, and x_p is relatively close to this: This is *antiphase* with $\sin(t)$ and has large amplitude. When ω_0 is just greater than 1, the same term occurs and dominates but now is a positive multiple of $\sin(t)$, so the system response is *in phase* with the input.

(d) This is a tricky question. When ω_0 is not an odd integer, the solution x_p above is periodic of period 2π. The general solution of the homogeneous equation is $a\cos(\omega_0 t) + b\sin(\omega_0 t)$, which is periodic of period $\frac{2\pi}{\omega_0}$. The sum is periodic if some multiple of 2π is equal to some multiple of $\frac{2\pi}{\omega_0}$, and this happens when ω_0 is a rational number (but not an odd integer).

(e) Yes. [They are periodic of period 2π if ω_0 is an even integer.]