The Phase Plane

The sort of system for which we will be trying to sketch solutions can be written in the form

$$\begin{array}{ll} x' &= ax + by \\ y' &= cd + dy \end{array} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}, \quad \text{where} \quad A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \quad (1)$$

where a, b, c, d are constants.

A **solution** of this system has the form (we write it two ways)

$$\mathbf{x}(t) = \left(\begin{array}{c} x(t) \\ y(t) \end{array} \right), \qquad \begin{array}{ll} x &= x(t) \\ y &= y(t). \end{array} \qquad (2)$$

It is a vector function of t whose components satisfy the system (1) when they are substituted in for x and y. In general, you learned in 18.02 and physics that such a vector function describes a motion in the xy-plane; the equations in (2) tell how the point (x, y) moves in the xy-plane as the time t varies. The moving point traces out a curve called the **trajectory** of the solution (2). The xy-plane itself is called the **phase plane** for the system (1). We show a sketch of a trajectory at right. Notice the arrow is used to indicate the direction of increasing time.
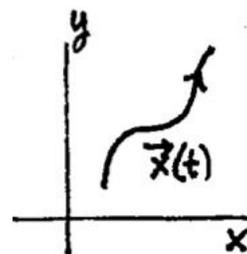
We use the term **phase portrait** to mean the graphs of enough trajectories to give a good sense of all the solutions to the system (1)

1. Critical Points

Definition. A critical point is a point where the derivatives are 0. Therefore a point (x_0, y_0) is a **critical point** of the system (1) if

$$\left(\begin{array}{c} x' \\ y' \end{array} \right) = A \left(\begin{array}{c} x_0 \\ y_0 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right).$$

The equations of the system (1) show this is equivalent to

$$x = x_0, \quad y = y_0 \qquad \text{is a (constant) solution to (1).}$$

Critical points are the key to our qualitative view of systems. We classify the linear systems by their behavior near critical points.

For the linear system constant coefficient system (1) there is always a critical point at $(0,0)$. If the matrix A is invertible then this is the only critical point.

2.   Sketching Principle

When sketching integral curves for direction fields we saw that integral curves did not cross. For the system (1) we have a similar principle.

Sketching Principle.   Two trajectories of (1) cannot intersect.

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