Answers to Problem Set Number 8 for 18.04. MIT (Fall 1999)

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Contents

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1 Problems from the book by Saff and Snider.

1.1 Problem 02 in section 5.6.

In general, we have:

- The order of pole of a function f (at a point $z = z_0$) is equal to the order of the zero of the function $\frac{1}{f}$ (at the same point).
- The order of a zero of g^n (at any point where g has a zero) is equal to n times the order of the zero of g. Here *n is a positive integer!*

To apply this result to this problem we just need to find the order of the zero (at $z = 0$) of

$$g(z) = 2\cos(z) - 2 + z^2 \,.$$

Using the Taylor expansion for $\cos(z)$ at $z = 0$, we find:

$$g(z) = -2 + z^2 + 2\cos(z) = -2 + z^2 + 2(1 - \frac{z^2}{2} + \frac{z^4}{24} + \ldots) = \frac{z^4}{12} + \ldots$$

Thus, the order of the zero of g at $z = 0$ is 4. It follows that f has a pole of order 8 at $z = 0$.

1.2 Problem 3b in section 5.6.

We know that the function $h(z) = \sin\left(\frac{1}{z}\right)$ has an essential singularity[1] at $z = 0$, thus the function $g(z) = \sin\left(\frac{1}{1-z}\right)$ has an essential singularity at $z = 1$. Consider now the function

$$f(z) = zg(z) = z\sin\left(\frac{1}{1-z}\right) \,.$$

It is clear that f has a single zero at $z = 0$ (since $g(0) \neq 0$), an essential singularity at $z = 1$ and no other singularity in the finite complex plane.

[1] **For any entire function $F = F(z)$ that is not a polynomial, $F(1/z)$ has an essential singularity at $z = 0$. Can you see why?**

1.3 Problem 08 in section 5.6.

We will show that $\cos\left(\frac{1}{z}\right)$ achieves every complex value c in the disk $|z| < \epsilon$, where ϵ is any positive constant. This will verify Picard's theorem in this case.

We begin by recalling that we can express $\cos^{-1}$ in terms of the logarithm (see equation 9 in section 3.3, page 92 of the book). Namely:

$$\cos^{-1}(c) = -i\,\log(c + \sqrt{c^2 - 1}) = -i\,\mathrm{Log}(c + \sqrt{c^2 - 1}) + 2n\pi i.$$

By picking a sufficiently large n, we can find a value w of $\cos^{-1}(c)$ such that $|w| > \epsilon^{-1}$. Then, for $z = \dfrac{1}{w}$ (which clearly satisfies $|z| < \epsilon$), we have:

$$\cos\left(\frac{1}{z}\right) = \cos(w) = \cos(\cos^{-1}(c)) = c\,.$$

Notice that in this calculation we pick one value for $\sqrt{c^2 - 1}$ (does not matter which) and stick with it throughout.

1.4 Problem 12 in section 5.6.

If f has a pole of order m at z_0, then we can write

$$f(z) = \frac{1}{(z - z_0)^m} h(z)\,,$$

where h is analytic at $z = z_0$ and $h(z_0) \neq 0$. Thus

$$\begin{aligned} g(z) &= \frac{f'(z)}{f(z)} \\ &= \frac{-m(z - z_0)^{-m-1}h(z) + (z - z_0)^{-m}h'(z)}{(z - z_0)^{-m}h(z)} \\ &= \frac{1}{z - z_0}\,\frac{-mh(z) + (z - z_0)h'(z)}{h(z)}\,. \end{aligned}$$

Let $j(z) = \dfrac{-mh(z) + (z - z_0)h'(z)}{h(z)}$. Then we have

- j is analytic at $z = z_0$, since $h(z_0) \neq 0$.
- $j(z_0) = \dfrac{-mh(z_0)}{h(z_0)} = -m \neq 0$.

Therefore j has no pole or zero at $z = z_0$. Thus $g(z) = \dfrac{1}{z - z_0} j(z)$ has a simple pole at $z = z_0$ and the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion of g is $g_{-1} = -m$; that is: the negative of the order of the pole.

1.5 Problem 04 in section 5.7.

Set $z = \dfrac{1}{w}$. By definition, $f(z)$ has an essential singularity at $z = \infty$ if $f(w)$ has an essential singularity at $w = 0$. Interpreting behavior of $f(z)$ at $z = \infty$ as behavior of $f(w)$ at $w = 0$, we get the following formulation for Picard's theorem at $z = \infty$:

If f is a function with essential singularity at ∞, then f assumes every complex value (with possibly one exception) in any open set that contains ∞ (i.e. any open set whose complement is bounded).

For $f(z) = e^z$, we need to show that for any $r > 0$, f assumes every complex value — except possibly one (obviously 0 in this case) — in the open set $N_r = \{|z| > r\}$. We do this next.

Given any complex number $c \neq 0$ and $r > 0$, we have:

$$\log(c) = \mathrm{Log}(c) + 2in\pi\,,$$

where $n \in \mathbf{Z}$. Since $\mathrm{Log}(c)$ is a constant, we can find n large enough so that

$$|\mathrm{Log}(c) + 2in\pi| \geq 2n\pi - |\mathrm{Log}(c)| > r.$$

Setting $z = \mathrm{Log}(c) + 2in\pi \in N_r$, we have $f(z) = c$. This finishes the verification of the theorem for this function.

1.6 Problem 04 in section 5.8.

Consider a function $f = f(z)$, analytic in a deleted neighborhood of $z = 0$, such that $f(\dfrac{1}{n}) = 0$ for all $n = \pm 1, \pm 2, \ldots$ Since f is analytic in a deleted neighborhood of $z = 0$, f has an isolated singularity at $z = 0$. There are then three possible cases:

1. f has removable singularity at $z = 0$. In this case, we can define $f(0) = c$ (for some c) so that f is analytic in the whole neighborhood of $z = 0$. However, by hypothesis, $f(\dfrac{1}{n}) = 0$ for all n. Thus f vanishes on the sequence $\{\dfrac{1}{n}\}$, which converges to 0. Then, by Theorem 22 (section 5.8 page 236), f is identically zero.

2. f has a pole at $z = 0$. Then, by Lemma 6 (section 5.6, page 221), $|f(z)| \to \infty$ as $z \to 0$. This cannot happen given the condition that $f(\dfrac{1}{n}) = 0$.

3. f has an essential singularity at $z = 0$. This is possible, for example: $f(z) = \sin(\frac{\pi}{z})$ satisfies this condition.

In summary: case 2 cannot happen and if case 1 happens, then f is identically zero. Therefore, either f **is identically zero or it has an essential singularity at** $z = 0$.

1.7 Problem 06 in section 6.1.

Following the same idea used in problem 5.6.12, write

$$f(z) = (z - z_0)^m h(z)\,,$$

where h is analytic at $z = z_0$ and $h(z_0) \neq 0$. Thus

$$g(z) = \frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{h'(z)}{h(z)}\,.$$

Since $h(z_0) \neq 0$, the second term on the right of this equality (i.e.: $\dfrac{h'(z)}{h(z)}$ is analytic at $z = z_0$). It follows that g has a simple pole at $z = z_0$, with residue:

$$\text{Res}(g; z_0) = m\,.$$

Notice that, in the case of a pole of order m, problem 5.6.12 tells us that $\text{Res}(g; z_0) = -m$. Thus $g = f'/f$ can be used to **count poles and zeros** via its residues.

1.8 Problem 07 in section 6.1.

Let $f(z) = e^{1/z}\sin(1/z)$. Then f has an essential singularity at $z = 0$ and is analytic everywhere else. Thus, to evaluate the requested integral, we just need to calculate the reside of f at $z = 0$.

The Laurent expansion for $f(z)$ at $z = 0$ is the product of the Laurent expansions for its two factors, namely:

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots \\ \sin(\frac{1}{z}) &= \frac{1}{z} - \frac{1}{6z^3} + \ldots \\ f(z) &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{3z^3} + \ldots \end{aligned}$$

Thus $\mathrm{Res}(f;0) = 1$, so that:

$$\int_{|z|=1} f(z) = 2\pi i \,\mathrm{Res}(f;0) = 2\pi i\,.$$

Note: The presumption here is that the (unit circle) path is being traced counterclockwise (otherwise the sign of the integral will change), though the problem statement in the book says nothing about this.

1.9 Problem 02 in section 6.2.

To evaluate

$$I = \int_0^\pi \frac{8d\theta}{5+2\cos\theta}\,,$$

we begin by using the fact that $\cos(\theta)$ is even and periodic (thus $\cos(\theta) = \cos(2\pi - \theta)$) to replace the integral by one over $[0, 2\pi]$. That is

$$I = \int_0^\pi \frac{8d\theta}{5+2\cos\theta} = \int_\pi^{2\pi} \frac{8d\theta}{5+2\cos\theta}\,,$$

so that:

$$I = \int_0^{2\pi} \frac{4d\theta}{5+2\cos\theta}.$$

Substitute now into this integral:

$$\cos\theta = \frac{1}{2}(z + \frac{1}{z}) \quad \text{and} \quad d\theta = \frac{dz}{iz}\,,$$

where $z = e^{i\theta}$. Then:

$$I = \int_C \frac{-4i dz}{z^2+5z+1}\,,$$

where the integration is over the unit circle (counter-clockwise) C. The two zeroes of the denominator $g(z) = z^2 + 5z + 1$ in the integrand are:

$$z_1 = \frac{-5+\sqrt{21}}{2} \quad \text{and} \quad z_2 = \frac{-5-\sqrt{21}}{2}\,.$$

Note now that z_2 lies outside C and z_1 lies inside C, with residue for $\frac{1}{g}$ at $z = z_1$ given by:

$$\mathrm{Res}(\frac{1}{g}; z_1) = \lim_{z\to z_1}(z - z_1)\frac{1}{(z-z_1)(z-z_2)} = \frac{1}{z_1 - z_2} = \frac{1}{\sqrt{21}}\,.$$

Now we evaluate the integral using residues:

$$I = -4i \times 2\pi i \times \mathrm{Res}(\frac{1}{g}; z_1) = \frac{8\pi}{\sqrt{21}}\,.$$

2 Other problems.

2.1 Problem 8.1 in 1999.

Statement.

Find three terms in the Laurent expansion for

$$f(z) = \frac{1}{\sin^2(z)}\,,$$

valid in the annulus $\pi < |z| < 2\pi$.

Note: these three terms could be the three "central" ones (i.e.: the ones proportional to z^n, with $n = -1$, $n = 0$ and $n = 1$); but any three will do.

Solution.

The entire function $\sin^2(z)$ has exactly three zeroes in the open disk $D = \{|z| < 2\pi\}$; namely at $z = 0$, $z = \pi$ and at $z = -\pi$. Thus $\dfrac{1}{\sin^2(z)}$ has exactly three poles (and no other singularities) in this region. The idea in computing the requested Laurent expansion is then:

1. First subtract the singular part of the function at the three poles, so as to get an analytic function in the open disk $D = \{|z| < 2\pi\}$.

2. Find the Taylor expansion for the analytic function in the open disk D, constructed in the first step.

3. Add (to the Taylor expansion constructed in the second step) the Laurent expansions for the singular parts.

In order to implement this program, we first need to find out what the singular parts at the poles are. We do this (and quite a bit more next).

Using the Taylor expansion for $\sin(z)$ at $z = 0$, we can write

$$\sin(z) = z(1 - \mu(z))\,, \quad \text{where} \quad \mu = \frac{1}{6}\,z^2 - \frac{1}{120}\,z^4 + \frac{1}{5040}\,z^6 + \dots \tag{2.1.1}$$

is entire and vanishes at $z = 0$. Thus[2]

$$\begin{aligned}\frac{1}{\sin(z)} &= \frac{1}{z(1-\mu)} = \frac{1}{z}(1+\mu+\mu^2+\ldots) = \frac{1}{z}(1+\frac{1}{6}\,z^2+\frac{7}{360}\,z^4+\frac{31}{15120}\,z^6+\ldots) \\ &= \frac{1}{z}+\frac{1}{6}\,z+\frac{7}{360}\,z^3+\frac{31}{15120}\,z^5+\ldots \end{aligned} \tag{2.1.2}$$

which **must be** the Laurent expansion for $\dfrac{1}{\sin(z)}$, valid on $0 < |z| < \pi$. Squaring this we get:

$$\frac{1}{\sin^2(z)} = \frac{1}{z^2}+\frac{1}{3}+\frac{1}{15}\,z^2+\frac{2}{189}\,z^4+\ldots\,, \tag{2.1.3}$$

which is the Laurent expansion for $\dfrac{1}{\sin^2(z)}$, valid on $0 < |z| < \pi$. Notice that

$$g(z) = \frac{1}{\sin^2(z)} - \frac{1}{z^2} = \frac{1}{3}+\frac{1}{15}\,z^2+\frac{2}{189}\,z^4+\ldots = \sum_{n=0}^{\infty} g_n\,z^n \tag{2.1.4}$$

is analytic on $|z| < \pi$. Because $\dfrac{1}{\sin^2(z)}$ is periodic (of period π), its Laurent expansions valid on $0 < |z \pm \pi| < \pi$ follow from (2.1.3) above (by the changes of variable $z \to z \pm \pi$).

It now follows that

$$h(z) = \frac{1}{\sin^2 z} - \frac{1}{(z+\pi)^2} - \frac{1}{(z-\pi)^2} - \frac{1}{z^2} \tag{2.1.5}$$

is analytic in the open disk $D = \{|z| < 2\pi\}$. Thus it has a Taylor expansion

$$h(z) = \sum_{n=0}^{\infty} h_n z^n\,, \tag{2.1.6}$$

that converges for $|z| < 2\pi$. **This completes the first step in the plan.**

We have

$$\begin{aligned}\frac{1}{(z-\pi)^2} &= -\frac{d}{dz}\,\frac{1}{z-\pi} = -\frac{d}{dz}\,\frac{1}{z}\,\frac{1}{1-\pi/z} = -\frac{d}{dz}\,\frac{1}{z}\sum_{n=0}^{\infty}\frac{\pi^n}{z^n} = -\frac{d}{dz}\sum_{n=1}^{\infty}\frac{\pi^{n-1}}{z^n} \\ &= \sum_{n=2}^{\infty}(n-1)\frac{\pi^{n-2}}{z^n} \quad \text{(valid on } |z| > \pi\text{)},\end{aligned} \tag{2.1.7}$$

with a similar expansion for $\dfrac{1}{(z+\pi)^2}$. Thus we can write (using (2.1.5) and (2.1.6)):

$$\frac{1}{\sin^2(z)} = \sum_{n=0}^{\infty} h_n z^n + \frac{1}{z^2} + \sum_{n=1}^{\infty}(4n-2)\frac{\pi^{2n-2}}{z^{2n}}\,, \tag{2.1.8}$$

[2] This is the kind of manipulation that can be made painless by using a symbolic program, such as Maple.

which converges for $\pi < |z| < 2\pi$ and thus provides the desired Laurent expansion. Notice that this last expansion converges for $\pi < |z| < 2\pi$ because (2.1.6) converges for $|z| < 2\pi$ while (2.1.7) converges for $|z| > \pi$.

We have now **completed the first and third steps in the plan.** As far as the answer to the stated problem, (2.1.8) is enough — since it provides far more than three terms in the requested Laurent expansion (all the negative terms, in fact). We will, however (for completeness) indicate now how to do the second step in the program; that is: **how do we find the coefficients h_n in (2.1.6)?**.

We begin by considering the Taylor expansions for $\dfrac{1}{(z \pm \pi)^2}$ valid near zero. Following the same idea that we used in (2.1.7), it is clear that:

$$\frac{1}{(z-\pi)^2} = \frac{d}{dz}\frac{1}{\pi}\frac{1}{1 - z/\pi} = \frac{d}{dz}\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{z^n}{\pi^n} = \sum_{n=0}^{\infty}(n+1)\frac{z^n}{\pi^{n+2}} \quad \text{(valid on } |z| < \pi), \tag{2.1.9}$$

with a similar expansion for $\dfrac{1}{(z+\pi)^2}$. Thus we can write (using (2.1.4) and (2.1.5)):

$$h(z) = g(z) - \frac{1}{(z+\pi)^2} - \frac{1}{(z-\pi)^2} = \sum_{n=0}^{\infty}(g_n - (n+1)\frac{2\sigma_n}{\pi^{n+2}})z^n\,, \tag{2.1.10}$$

where $\sigma_n = 1$ for n even and $\sigma_n = 0$ for n odd. Now, this **must** be the Taylor expansion for $h(z)$ valid on $|z| < 2\pi$ (i.e.: (2.1.6)). Why? Well, since the Taylor expansion for $g(z)$ in (2.1.4) and the Taylor expansions for $\dfrac{1}{(z \pm \pi)^2}$ in (2.1.9) all converge for $|z| < \pi$, we know that (2.1.10) will converge for at least $|z| < \pi$. This means that it must be the Taylor series for $h(z)$ centered at $z = 0$ — and we know this series must converge in a disk reaching out to the nearest singularities at $z = \pm 2\pi$.

This finishes the problem. Using (2.1.1) and (2.1.2) we can compute the coefficients g_n in (2.1.4) up to any desired order (though the calculations get hairy for n large). Then (2.1.8) gives the requested Laurent expansion, with $h_n = g_n - 2\,\sigma_n\,(n+1)\,\pi^{-n-2}$ — as given by (2.1.10).

THE END.