18.04 Recitation 5
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1. Let $T(x, y)$ be the steady state temperature distribution on a square metal plate, where $(x, y) \in [0, 1] \times [0, 1]$. Such a distribution is known to be a harmonic function. Suppose the edges of the square have the following temperature distributions:

- Bottom: $T(x, 0) = 100x^2$
- Top: $T(x, 1) = 100x^2 + 100$
- Left: $T(0, y) = 100y^2$
- Right: $T(1, y) = 100y^2 + 100$

What are the maximum and minimum temperatures on the plate?

Ans: By the maximum principle, the maximum and mimum occur on the boundary of the plate. Let us compute the maximum and minimum on all four regions of the boundary.

- Bottom: $\max = 100, \min = 0$
- Top: $\max = 200, \min = 100$
- Left: $\max = 100, \min = 0$
- Right: $\max = 200, \min = 100$

This shows that the global maximum is 200 and the global minimum is 0.

2. Show that $u = \sin(x) \cosh(y)$ is harmonic. Find a harmonic conjugate.

Ans: We check that u satisfies Laplace's equation. Indeed,

$$\begin{aligned} u_x &= \cos(x) \cosh(y) \\ u_{xx} &= -\sin(x) \cosh(y) \\ u_y &= \sin(x) \sinh(y) \\ u_{yy} &= \sin(x) \cosh(y) \end{aligned}$$

Therefore, $u_{xx} + u_{yy} = 0$.

To find a harmonic conjugate, suppose $f(x, y) = u(x, y) + iv(x, y)$ were an analytic function. Then, we must have

$$\begin{aligned} f' &= u_x + iv_x \\ &= u_x - iu_y \\ &= \cos(x) \cosh(y) - i \sin(x) \sinh(y) \\ &= \cos(x + iy) \end{aligned}$$

This suggests that $f(x, y)$ should be $\sin(x + iy)$, and indeed, note that $u(x, y) = \mathrm{Re}(\sin(x + iy))$.