Manipulating Continuous Random Variables
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1 Learning Goals

1. Be able to find the pdf and cdf of a random variable defined in terms of a random variable with known pdf and cdf.

2 Transformations of Random Variables

If $Y = aX+b$ then the properties of expectation and variance tell us that $E(Y) = aE(X)+b$ and $\mathrm{Var}(Y) = a^2\mathrm{Var}(X)$. But what is the distribution function of Y? If Y is continuous, what is its pdf?

Often, when looking at transforms of discrete random variables we work with tables. For continuous random variables transforming the pdf is just change of variables ('u-substitution') from calculus. Transforming the cdf makes direct use of the definition of the cdf.

Let's remind ourselves of the basics:

1. The cdf of X is $F_X(x) = P(X \le x)$.
2. The pdf of X is related to F_X by $f_X(x) = F_X'(x)$.

Example 1. Let $X \sim U(0,2)$, so $f_X(x) = 1/2$ and $F_X(x) = x/2$ on [0,2]. What is the range, pdf and cdf of $Y = X^2$?

answer: The range is easy: $[0,4]$.
To find the cdf we work systematically from the definition.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) = \sqrt{y}/2.$$

To find the pdf we can just differentiate the cdf

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \boxed{\frac{1}{4\sqrt{y}}}.$$

An alternative way to find the pdf directly is by change of variables. The trick here is to remember that it is $f_X(x)dx$ which gives probability ($f_X(x)$ by itself is probability density). Here is how the calculation goes in this example.

$$y = x^2 \;\Rightarrow\; dy = 2x\,dx \;\Rightarrow\; dx = \frac{dy}{2\sqrt{y}}$$

$$f_X(x)\,dx = \frac{dx}{2} = \frac{dy}{4\sqrt{y}} = f_Y(y)\,dy$$

Therefore $\boxed{f_Y(y) = \frac{dy}{4\sqrt{y}}}$

Example 2. Let $X \sim \exp(\lambda)$, so $f_X(x) = \lambda e^{-\lambda x}$ on $[0, \infty]$. What is the density of $Y = X^2$?

answer: Let's do this using the change of variables.

$$y = x^2 \;\Rightarrow\; dy = 2x\,dx \;\Rightarrow\; dx = \frac{dy}{2\sqrt{y}}$$

$$f_X(x)\,dx = \lambda e^{-\lambda x}\,dx = \lambda e^{-\lambda\sqrt{y}}\,\frac{dy}{2\sqrt{y}} = f_Y(y)\,dy$$

Therefore $\boxed{f_Y(y) = \frac{\lambda}{2\sqrt{y}}\,e^{-\lambda\sqrt{y}}}$.

Example 3. Assume $X \sim N(5, 3^2)$. Show that $Z = \frac{X-5}{3}$ is standard normal, i.e., $Z \sim N(0, 1)$.

answer: Again using the change of variables and the formula for $f_X(x)$ we have

$$z = \frac{x-5}{3} \;\Rightarrow\; dz = \frac{dx}{3} \;\Rightarrow\; dx = 3\,dz$$

$$f_X(x)\,dx = \frac{1}{3\sqrt{2\pi}} e^{-(x-5)^2/(2\cdot 3^2)}\,dx = \frac{1}{3\sqrt{2\pi}} e^{-z^2/2}\,3\,dz = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = f_Z(z)\,dz$$

Therefore $f_Z(z) = \frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}$. Since this is exactly the density for $N(0, 1)$ we have shown that Z is standard normal.

This example shows an important general property of normal random variables which we give in the next example.

Example 4. Assume $X \sim N(\mu, \sigma^2)$. Show that $Z = \frac{X-\mu}{\sigma}$ is standard normal, i.e., $Z \sim N(0, 1)$.

answer: This is exactly the same computation as the previous example with μ replacing 5 and σ replacing 3. We show the computation without comment.

$$z = \frac{x-\mu}{\sigma} \;\Rightarrow\; dz = \frac{dx}{\sigma} \;\Rightarrow\; dx = \sigma\,dz$$

$$f_X(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\cdot\sigma^2)}\,dx = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}\,\sigma\,dz = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = f_Z(z)\,dz$$

Therefore $f_Z(z) = \frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}$. This shows Z is standard normal.