Differential equations and e^{At}

The system of equations below describes how the values of variables u_1 and u_2 affect each other over time:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$
$$\frac{du_2}{dt} = u_1 - 2u_2.$$

Just as we applied linear algebra to solve a difference equation, we can use it to solve this differential equation. For example, the initial condition $u_1 = 1$, $u_2 = 0$ can be written $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Differential equations $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$

By looking at the equations above, we might guess that over time u_1 will decrease. We can get the same sort of information more safely by looking at the eigenvalues of the matrix $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$ of our system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$. Because A is singular and its trace is -3 we know that its eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -3$. The solution will turn out to include e^{-3t} and e^{0t}. As t increases, e^{-3t} vanishes and $e^{0t} = 1$ remains constant. Eigenvalues equal to zero have eigenvectors that are *steady state* solutions.

$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector for which $A\mathbf{x}_1 = 0\mathbf{x}_1$. To find an eigenvector corresponding to $\lambda_2 = -3$ we solve $(A - \lambda_2 I)\mathbf{x}_2 = \mathbf{0}$:

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_2 = 0 \quad \text{so} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and we can check that $A\mathbf{x}_2 = -3\mathbf{x}_2$. The general solution to this system of differential equations will be:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2.$$

Is $e^{\lambda_1 t}\mathbf{x}_1$ really a solution to $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$? To find out, plug in $\mathbf{u} = e^{\lambda_1 t}\mathbf{x}_1$:

$$\frac{d\mathbf{u}}{dt} = \lambda_1 e^{\lambda_1 t}\mathbf{x}_1,$$

which agrees with:

$$A\mathbf{u} = e^{\lambda_1 t} A\mathbf{x}_1 = \lambda_1 e^{\lambda_1 t}\mathbf{x}_1.$$

The two "pure" terms $e^{\lambda_1 t}\mathbf{x}_1$ and $e^{\lambda_2 t}\mathbf{x}_2$ are analogous to the terms $\lambda_i^k \mathbf{x}_i$ we saw in the solution $c_1\lambda_1^k\mathbf{x}_1 + c_2\lambda_2^k\mathbf{x}_2 + \cdots + c_n\lambda_n^k\mathbf{x}_n$ to the difference equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

Plugging in the values of the eigenvectors, we get:

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We know $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so at $t = 0$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$c_1 = c_2 = 1/3$ and $\mathbf{u}(t) = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

This tells us that the system starts with $u_1 = 1$ and $u_2 = 0$ but that as t approaches infinity, u_1 decays to $2/3$ and u_2 increases to $1/3$. This might describe stuff moving from u_1 to u_2.

The steady state of this system is $\mathbf{u}(\infty) = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

Stability

Not all systems have a steady state. The eigenvalues of A will tell us what sort of solutions to expect:

1. Stability: $\mathbf{u}(t) \to 0$ when $\mathrm{Re}(\lambda) < 0$.
2. Steady state: One eigenvalue is 0 and all other eigenvalues have negative real part.
3. Blow up: if $\mathrm{Re}(\lambda) > 0$ for any eigenvalue λ.

If a two by two matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two eigenvalues with negative real part, its trace $a + d$ is negative. The converse is not true: $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ has negative trace but one of its eigenvalues is 1 and e^{1t} blows up. If A has a positive determinant and negative trace then the corresponding solutions must be stable.

Applying S

The final step of our solution to the system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ was to solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In matrix form:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

or $S\mathbf{c} = \mathbf{u}(0)$, where S is the eigenvector matrix. The components of $\mathbf{c}$ determine the contribution from each pure exponential solution, based on the initial conditions of the system.

In the equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$, the matrix A couples the pure solutions. We set $\mathbf{u} = S\mathbf{v}$, where S is the matrix of eigenvectors of A, to get:

$$S\frac{d\mathbf{v}}{dt} = AS\mathbf{v}$$

or:

$$\frac{d\mathbf{v}}{dt} = S^{-1}AS\mathbf{v} = \Lambda\mathbf{v}.$$

This diagonalizes the system: $\frac{dv_i}{dt} = \lambda_i v_i$. The general solution is then:

$$\begin{aligned} \mathbf{v}(t) &= e^{\Lambda t}\mathbf{v}(0), \qquad \text{and} \\ \mathbf{u}(t) &= Se^{\Lambda t}S^{-1}\mathbf{v}(0) = e^{At}\mathbf{u}(0). \end{aligned}$$

Matrix exponential e^{At}

What does e^{At} mean if A is a matrix? We know that for a real number x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots.$$

We can use the same formula to define e^{At}:

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots.$$

Similarly, if the eigenvalues of At are small, we can use the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to estimate $(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots$.

We've said that $e^{At} = Se^{\Lambda t}S^{-1}$. If A has n independent eigenvectors we can prove this from the definition of e^{At} by using the formula $A = S\Lambda S^{-1}$:

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots \\ &= SS^{-1} + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}}{2}t^2 + \frac{S\Lambda^3 S^{-1}}{6}t^3 + \cdots \\ &= Se^{\Lambda t}S^{-1}. \end{aligned}$$

It's impractical to add up infinitely many matrices. Fortunately, there is an easier way to compute $e^{\Lambda t}$. Remember that:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

When we plug this in to our formula for e^{At} we find that:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}.$$

This is another way to see the relationship between the stability of $\mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{v}(0)$ and the eigenvalues of A.

Second order

We can change the second order equation $y'' + by' + ky = 0$ into a two by two first order system using a method similar to the one we used to find a formula for the Fibonacci numbers. If $u = \begin{bmatrix} y' \\ y \end{bmatrix}$, then

$$u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}.$$

We could use the methods we just learned to solve this system, and that would give us a solution to the second order scalar equation we started with.

If we start with a kth order equation we get a k by k matrix with coefficients of the equation in the first row and 1's on a diagonal below that; the rest of the entries are 0.