# Exercises on symmetric matrices and positive definiteness

**Problem 25.1:** (6.4 #10. *Introduction to Linear Algebra:* Strang) Here is a quick "proof" that the eigenvalues of all real matrices are real:

**False Proof:** $Ax = \lambda x$ gives $x^T Ax = \lambda x^T x$ so $\lambda = \dfrac{x^T Ax}{x^T x}$ is real.

There is a hidden assumption in this proof which is not justified. Find the flaw by testing each step on the $90°$ rotation matrix:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $\lambda = i$ and $\mathbf{x} = (i, 1)$.

**Solution:** We can esily confirm that $Ax = \lambda x = \begin{bmatrix} -1 \\ i \end{bmatrix}$. Next, check if $\mathbf{x}^T Ax = \lambda \mathbf{x}^T \mathbf{x}$ is true for the $90°$ rotation matrix:

$$\mathbf{x}^T Ax = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} -1 \\ i \end{bmatrix} = 0$$

$$\lambda \mathbf{x}^T \mathbf{x} = i \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0$$

$$\mathbf{x}^T Ax = \lambda \mathbf{x}^T \mathbf{x}. \checkmark$$

Note that $\mathbf{x}^T \mathbf{x} = 0$. Since the next and last step involves dividing by this term, the hidden assumption must be that $\mathbf{x}^T \mathbf{x} \neq 0$. If $x = (a, b)$ then

$$\mathbf{x}^T \mathbf{x} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2.$$

The "proof" assumes that the squares of the components of the eigenvector cannot sum to zero: $a^2 + b^2 \neq 0$. This may be false if the components are complex.

**Problem 25.2:** (6.5 #32.) A *group* of nonsingular matrices includes $AB$ and $A^{-1}$ if it includes $A$ and $B$. "Products and inverses stay in the group." Which of these are groups?

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a) Positive definite symmetric matrices $A$.

b) Orthogonal matrices $Q$.

c) All exponentials $e^{tA}$ of a fixed matrix $A$.

d) Matrices $D$ with determinant 1.

**Solution:**

a) The positive definite symmetric matrices $A$ **do not form a group.** To show this, we provide a counterexample in the form of two positive definite symmetric matrices $A$ and $B$ whose product is not a positive definite symmetric matrix.

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ then $AB = \begin{bmatrix} 2.5 & 2 \\ 1.5 & 1.5 \end{bmatrix}$ is not symmetric.

b) The orthogonal matrices $Q$ **form a group**. If $A$ and $B$ are orthogonal matrices, then:

$$A^T A = I \Rightarrow A^{-1} = A^T \Rightarrow A^{-1} \text{ is orthogonal, and}$$
$$B^T B = I \Rightarrow (AB)^T AB = B^T A^T AB = B^T B = I \Rightarrow AB \text{ is orthogonal.}$$

c) The exponentials $e^{tA}$ of a fixed matrix $A$ **form a group.** For the elements $e^{pA}$ and $e^{qA}$:

$$(e^{pA})^{-1} = e^{-pA} \text{ is of the form } e^{tA}$$
$$e^{pA} e^{qA} = e^{(p+q)A} \text{ is of the form } e^{tA}$$

d) The matrices $D$ with determinant 1 **form a group.** If $\det A = 1$ then $\det A^{-1} = 1$. If matrices $A$ and $B$ have determinant 1 then their product also has determinant 1:

$$\det(AB) = \det(A)\det(B) = 1.$$

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