SOLUTION SET X FOR 18.075–FALL 2004

5. Boundary-Value Problems and Characteristic-Function Representations

Please do Problems 49, 50, 58 and 61 first. Problems 54 and 55 have been removed from the original list of problems. Problems 36–38 and 60 are optional yet useful; the corresponding material of Sec. 5.8 will be discussed in class.

5.8. **Boundary-Value Problems Involving Nonhomogeneous Differential Equations.**

36. Obtain the solution of the problem $\frac{d^2y}{dx^2} + \Lambda y = h(x)$, $y(0) = y(l) = 0$ in the form

$$y(x) = \sum_{n=1}^{\infty} \frac{A_n}{\Lambda - n^2\pi^2/l^2} \sin \frac{n\pi x}{l} \ (0 \le x \le l)$$

when $\Lambda \neq (n\pi/l)^2$ $(n = 1, 2, \ldots)$, where A_n is the nth coeffient in the expansion $h(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$ $(0 < x < l)$, assuming that $h(x)$ is piecewise differentiable in $(0, l)$.

Solution. The strategy is to express the solution as $y = \sum a_n\varphi_n(x)$ where $\varphi_n(x)$ are the characteristic functions of the homogeneous equation $\frac{d^2y}{dx^2} + \lambda y = 0$. Then $(\Lambda - \lambda_n)a_n = A_n$ where A_n are the coefficients of $h(x) = \sum A_n\varphi_n(x)$.

First we will solve the homogeneous equation $\frac{d^2y}{dx^2} + \lambda y = 0,\ y(0) = y(l) = 0$. So

$$\begin{aligned}
y(x) &= C_1 \sin(\sqrt{\lambda}\cdot x) + C_2 \cos(\sqrt{\lambda}\cdot x) \\
\text{and } y(0) = 0 &\Rightarrow C_2 = 0 \\
\text{and } y(l) = 0 &\Rightarrow C_1 \sin(\sqrt{\lambda}\cdot x) = 0 \\
&\Rightarrow \sin(\sqrt{\lambda}\cdot x) = 0 \\
&\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2 \\
&\Rightarrow \varphi_n(x) = \sin\left(\frac{n\pi}{l}x\right)
\end{aligned}$$

So since we're given that $\Lambda \neq \left(\frac{n\pi}{l}\right)^2$, $n > 0$, this means that $\Lambda \neq \lambda_n$ so $a_n = \frac{A_n}{\Lambda - \lambda_n}$, $n > 0$ and the A_n are the coefficients in the expresion of $h(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$, $0 < x < l$. Thus, indeed in this case

$$y(x) = \sum a_n\varphi_n(x) = \sum_{n=1}^{\infty} \frac{A_n}{\Lambda - n^2\pi^2/l^2} \sin\left(\frac{n\pi x}{l}\right) \qquad (0 \le x \le l).$$

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37. Obtain the condition which must be satisfied by a piecewise differentiable function $h(x)$ in order that the problem $\frac{d^2y}{dx^2} + \frac{p^2\pi^2}{l^2}y = h(x),\ y(0) = y(l) = 0$ possesses a solution, when p is a positive integer, and express the corresponding most general solution as a series $\sum a_n \sin(n\pi x/l)$.

Solution. $\Lambda = \frac{p^2\pi^2}{l^2} = \lambda_p$ so since $(\Lambda - \lambda_n)a_n = A_n$ and $\Lambda = \lambda_p \Rightarrow A_p = 0$. But $A_p = \int_0^l h(x)\varphi_p(x)\ dx$. Hence, $\int_0^l h(x)\sin\left(\frac{p\pi x}{l}\right)\ dx = 0$. Thus, substituting into the solution of Problem 36,

$$\begin{aligned} y(x) &= \sum_{\substack{n=1\\ n\neq p}}^{\infty} \frac{A_n}{\frac{p^2\pi^2}{l^2} - \frac{n^2\pi^2}{l^2}} \cdot \sin\left(\frac{n\pi x}{l}\right) \\ &= \sum_{\substack{n=1\\ n\neq p}}^{\infty} \frac{l^2}{\pi^2}\frac{A_n}{p^2 - n^2}\sin\left(\frac{n\pi x}{l}\right) \\ &= \frac{l^2}{\pi^2}\sum_{\substack{n=1\\ n\neq p}}^{\infty} \frac{A_n}{p^2 - n^2}\sin\left(\frac{n\pi x}{l}\right) \qquad \text{(knowing that } A_p = 0.) \end{aligned}$$

38. Let $h(x) = \sin(p\pi x/l)$ in Problem 36, where p is a positive integer.

(a) Show that the solution of that problem then is $y = \frac{\sin(p\pi x/l)}{\Lambda - (p^2\pi^2/l^2)}$ if $\Lambda \neq (n\pi/l)^2$ $(n = 1, 2, \ldots)$.

(b) If $\Lambda = (r\pi/l)^2$, where r is a positive integer but $r \neq p$, show that the solution of part (a) becomes $y = \frac{l^2}{\pi^2}\frac{\sin(p\pi x/l)}{r^2 - p^2}$ and also show that to this solution can be added any constant multiple of $\sin(r\pi x/l)$.

(c) Account for the nonuniqueness of the solution in part (b).

Solution. (a) Let $h(x) = \sin\left(\frac{p\pi x}{l}\right)$, but $h(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$, so $A_n = 0$ except for $n = p$ and $A_p = 1$. So

$$y(x) = \frac{1}{\Lambda - \frac{p^2\pi^2}{l^2}} \cdot \sin\left(\frac{n\pi x}{l}\right) \text{ if } \Lambda \neq \left(\frac{n\pi x}{l}\right)^2,\ n > 0.$$

(b) Substituting in $\Lambda = \left(\frac{r\pi}{l}\right)^2$ in Problem 36, we get

$$y = \frac{l^2}{\pi^2} \cdot \frac{\sin\left(\frac{p\pi x}{l}\right)}{r^2 - p^2}$$

Now let

$$y = \frac{l^2}{\pi^2} \cdot \frac{\sin\left(\frac{p\pi x}{l}\right)}{r^2 - p^2} + C\sin\left(\frac{r\pi x}{l}\right), \quad C: \text{ const.},$$

and show that y satisfies $y'' + \Lambda y = h(x)$. Indeed,

$$\begin{aligned}
y' &= \frac{lp}{\pi(r^2-p^2)}\cos\left(\frac{p\pi x}{l}\right) + C\,\frac{r\pi}{l}\cos\left(\frac{r\pi x}{l}\right)\\
y'' &= -\frac{p^2}{r^2-p^2}\sin\left(\frac{p\pi x}{l}\right) - C\,\frac{r^2\pi^2}{l^2}\sin\left(\frac{r\pi x}{l}\right)\\
\text{So } y''+\Lambda y &= -\frac{p^2}{r^2-p^2}\sin\left(\frac{p\pi x}{l}\right) - C\,\frac{r^2\pi^2}{l^2}\sin\left(\frac{r\pi x}{l}\right)\\
&\quad +\frac{r^2\pi^2}{l^2}\left(\frac{l^2}{\pi^2(r^2-p^2)}\sin\left(\frac{p\pi x}{l}\right) + C\,\sin\left(\frac{r\pi x}{l}\right)\right)\\
&= -\frac{p^2}{r^2-p^2}\sin\left(\frac{p\pi x}{l}\right) + \frac{r^2}{r^2-p^2}\sin\left(\frac{p\pi x}{l}\right)\\
&= \sin\left(\frac{p\pi x}{l}\right) = h(x).
\end{aligned}$$

(c) The nonuniqueness of the solution is accounted for by remembering that the general solution of a second order nonhomogeneous differential equation is of the form $y = y_{\text{particular}} + y_{\text{homogeneous}}$. Now in the first part of (b) we found a particular solution, so we should be able to add the homogeneous solution to it and still be a solution. But from the solution of Problem 36, the homogeneous solution is $y_{\text{homogeneous}} = C_1\sin\left(\frac{r\pi x}{l}\right)$. Thus the general solution is

$$\begin{aligned}
y &= y_{\text{particular}} + y_{\text{homogeneous}}\\
&= \frac{l^2}{\pi^2}\cdot\frac{\sin\left(\frac{p\pi x}{l}\right)}{r^2-p^2} + C_1\sin\left(\frac{r\pi x}{l}\right)
\end{aligned}$$

which is exactly what we discovered in the second part of (b).

5.10. **Fourier Sine Series and Cosine Series.**

In almost all problems of this section, the main idea is to apply integration by parts in order to carry out the integrations for the coefficients of the Fourier series. *You are strongly advised to actually do the calculations at least once by yourselves.*

49. Expand each of the following functions in a Fourier sine series of period $2l$, over the interval $(0, l)$, and in each case sketch the function represented by the series in the interval $(-3l, 3l)$:

$(a)\ f(x) = \begin{cases} 0 & (x<0), \\ x(l-x) & (x>0), \end{cases}$ $\quad (b)\ f(x) = \begin{cases} 0 & (x<0), \\ x & (0<x<\frac{l}{2}), \\ l-x & (x>\frac{l}{2}), \end{cases}$

$(c)\ f(x) = \begin{cases} 1 & (x<\frac{l}{2}), \\ 0 & (x>\frac{l}{2}), \end{cases}$ $\quad (d)\ f(x) = \sin\frac{\pi x}{2l},$

$(e)\ f(x) = \begin{cases} \sin\frac{\pi x}{l} & (0<x<\frac{l}{2}), \\ 0 & (\text{otherwise}), \end{cases}$ $\quad (f)\ f(x) = \begin{cases} \frac{1}{\epsilon} & (0<x<\epsilon<l), \\ 0 & (\text{otherwise}), \end{cases}$

Solution. (a) Let $f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$ in $(0,l)$. Then we apply integration by parts to get

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\ dx \\ &= \frac{2}{l}\cdot\frac{-l}{n\pi}\int_0^l x(l-x)\ d\cos\frac{n\pi x}{l} \\ &= -\frac{2}{n\pi}\left[x(l-x)\cos\frac{n\pi x}{l}\Big|_0^l - \int_0^l \left(\cos\frac{n\pi x}{l}\right)\cdot(l-2x)\ dx\right] \\ &= \frac{2}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}\ dx \\ &= \frac{2}{n\pi}\cdot\frac{l}{n\pi}\int_0^l (l-2x)\ d\sin\frac{n\pi x}{l} \\ &= \frac{2l}{n^2\pi^2}\left[(l-2x)\sin\frac{n\pi x}{l}\Big|_0^l + \int_0^l 2\sin\frac{n\pi x}{l}\ dx\right] \\ &= \frac{4l}{n^2\pi^2}\int_0^l \sin\frac{n\pi x}{l}\ dx \\ &= \frac{4l^2}{n^3\pi^3}\left(-\cos\frac{n\pi x}{l}\right)\Big|_0^l \\ &= \frac{4l^2}{n^3\pi^3}[1-(-1)^n] \end{aligned}$$

It follows that

$$A_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the Fourier sine series reads as

$$f(x) = \frac{8l^2}{\pi^3}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\right) \quad \text{in } (0,l).$$

(b) Let $f(x)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$ in $(0,l)$. Then we get

$$\begin{aligned}
A_n &= \frac{2}{l}\left(\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right) \qquad (y=l-x)\\
&= \frac{2}{l}\left(\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{0} y\,\sin\frac{n\pi(l-y)}{l}\,d(l-y)\right)\\
&= \frac{2}{l}\left(\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx+(-1)^{n+1}\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx\right)\\
&= \frac{2\left(1+(-1)^{n+1}\right)}{l}\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx\\
&= \frac{-2\left(1+(-1)^{n+1}\right)}{n\pi}\int_0^{\frac{l}{2}} x\,d\cos\frac{n\pi x}{l}\\
&= \frac{-2\left(1+(-1)^{n+1}\right)}{n\pi}\left(x\cos\frac{n\pi x}{l}\Big|_0^{\frac{l}{2}}-\int_0^{\frac{l}{2}}\cos\frac{n\pi x}{l}\,dx\right)\\
&= \frac{2\left(1+(-1)^{n+1}\right)}{n\pi}\left(\int_0^{\frac{l}{2}}\cos\frac{n\pi x}{l}\,dx-\frac{l}{2}\cos\frac{n\pi}{2}\right)\\
&= \frac{2\left(1+(-1)^{n+1}\right)l}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{2\left(1+(-1)^{n+1}\right)}{n\pi}\cdot\frac{l}{2}\cdot\cos\frac{n\pi}{2}
\end{aligned}$$

So,

$$A_n=\begin{cases}0, & \text{if } n \text{ is even}\\ \frac{4l}{n^2\pi^2}\cdot(-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.}\end{cases}$$

The desired Fourier sine series reads as

$$f(x)=\frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l} \quad \text{in } (0,l).$$

(c) Let $f(x)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$ in $(0,l)$. Then we have

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^{\frac{l}{2}}\sin\frac{n\pi x}{l}\,dx\\
&= -\frac{2}{n\pi}\left(\cos x\big|_0^{\frac{n\pi}{2}}\right)\\
&= \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)=\frac{4}{n\pi}\sin^2\frac{n\pi}{4}.
\end{aligned}$$

Then,

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2\frac{n\pi}{4}}{n}\ \sin\frac{n\pi x}{l}.$$

(d) Let $f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$ in $(0, l)$.

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l \sin\frac{\pi x}{2l}\sin\frac{n\pi x}{l}\ dx \\
&= \frac{1}{l}\int_0^l \left(\cos\frac{(2n-1)\pi x}{2l} - \cos\frac{(2n+1)\pi x}{2l}\right)\ dx \\
&= \frac{2}{(2n-1)\pi}\sin x\Big|_0^{\frac{(2n-1)\pi}{2}} - \frac{2}{(2n+1)\pi}\sin x\Big|_0^{\frac{(2n+1)\pi}{2}} \\
&= (-1)^{n-1}\frac{8n}{(4n^2-1)\pi},
\end{aligned}$$

by use of the identities $\sin(n\pi-\frac{\pi}{2})-\sin(n\pi+\frac{\pi}{2}) = 2(-1)^{n+1}$ and $\sin(n\pi-\frac{\pi}{2})+\sin(n\pi+\frac{\pi}{2}) = 0$. Then

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot n}{4n^2-1}\ \sin\frac{n\pi x}{l}\quad \text{in}\quad (0, l).$$

(e) Let $f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$ in $(0, l)$. Then we have, for $\underline{n > 1}$,

$$\begin{aligned}
A_{n>1} &= \frac{2}{l}\int_0^{\frac{l}{2}} \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\ dx \\
&= \frac{1}{l}\int_0^{\frac{l}{2}} \left(\cos\frac{(n-1)\pi x}{l} - \cos\frac{(n+1)\pi x}{l}\right)\ dx \\
&= \frac{1}{(n-1)\pi}\sin x\Big|_0^{\frac{(n-1)\pi}{2}} - \frac{1}{(n+1)\pi}\sin x\Big|_0^{\frac{(n+1)\pi}{2}} \\
&= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}-1}\cdot\frac{2n}{\pi(n^2-1)}, & \text{if } n \text{ is even.}\end{cases}
\end{aligned}$$

On the other hand, for $n = 1$,

$$\begin{aligned}
A_1 &= \frac{2}{l}\int_0^{\frac{l}{2}} \left(\sin\frac{\pi x}{l}\right)^2\ dx \\
&= \frac{1}{l}\int_0^{\frac{l}{2}} \left(1 - \cos\frac{2\pi x}{l}\right)\ dx \\
&= \frac{1}{2}
\end{aligned}$$

Finally, we get the sine series

$$f(x) = \frac{1}{2}\sin\frac{\pi x}{l} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cdot n}{4n^2-1}\ \sin\frac{2n\pi x}{l}.$$

(f) Let $f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$ in $(0, l)$. Accordingly,

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^{\epsilon}\frac{1}{\epsilon}\sin\frac{n\pi x}{l}\ dx \\ &= -\frac{2}{\epsilon n\pi}\ \cos x\Big|_0^{\frac{n\pi\epsilon}{l}} \\ &= \frac{2}{\epsilon n\pi}\left(1-\cos\frac{n\pi\epsilon}{l}\right). \end{aligned}$$

Then

$$f(x) = \frac{2}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{1-\cos\frac{n\pi\epsilon}{l}}{n}\ \sin\frac{n\pi x}{l}.$$

50. (a-f) Expand each of the functions listed in Problem 49 in a Fourier cosine series of period $2l$, over the interval $(0, l)$, and in each case sketch the function represented by the series in the interval $(-3l, 3l)$.

Solution. (a) Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ in $(0, l)$. Accordingly,

$$\begin{aligned} A_0 &= \frac{1}{l}\int_0^{l} f(x)\ dx \\ &= \frac{1}{l}\int_0^{l} x(l-x)\ dx \\ &= \frac{1}{l}\left(l\,\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_{x=0}^{l} \\ &= \frac{l^2}{6}. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\ dx \\
&= \frac{2}{l}\int_0^l x(l-x)\cos\frac{n\pi x}{l}\ dx \\
&= \frac{2}{n\pi}\int_0^l x(l-x)\ d\left(\sin\frac{n\pi x}{l}\right) \\
&= -\frac{2}{n\pi}\int_0^l (l-2x)\sin\frac{n\pi x}{l}\ dx \\
&= \frac{2l}{n^2\pi^2}\int_0^l (l-2x)\ d\left(\cos\frac{n\pi x}{l}\right) \\
&= \frac{2l}{n^2\pi^2}\left((l-2x)\cos\frac{n\pi x}{l}\Big|_0^l + 2\int_0^l \cos\frac{n\pi x}{l}\ dx\right) \\
&= \frac{2l}{n^2\pi^2}[-l\ \cos(n\pi) - l\cos 0 + 2\cdot 0] \\
&= \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\frac{4l^2}{n^2\pi^2}, & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

It follows that

$$f(x) = \frac{l^2}{6} - \sum_{n=1}^{\infty}\frac{4l^2}{(2n)^2\pi^2}\cos\frac{2n\pi x}{l} = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi x}{l} \qquad \text{in } (0,l).$$

(b) Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ in $(0,l)$. Accordingly,

$$\begin{aligned}
A_0 &= \frac{1}{l}\left(\int_0^{\frac{l}{2}} x\ dx + \int_{\frac{l}{2}}^{l}(l-x)\ dx\right) \\
&= \frac{l}{4}
\end{aligned}$$

For $\underline{n \geq 1}$,

$$\begin{aligned}
A_n &= \frac{2}{l}\left(\int_0^{\frac{l}{2}} x\cos\frac{n\pi x}{l}\ dx + \int_{\frac{l}{2}}^{l} (l-x)\cos\frac{n\pi x}{l}\ dx\right) \qquad (y = l-x)\\
&= \frac{2}{l}\left[\int_0^{l/2} x\ \cos\frac{n\pi x}{l}\, dx + \int_{l/2}^{0} y\ \cos\frac{n\pi(l-y)}{l}\, dy\right]\\
&= \frac{2(1+(-1)^n)}{l}\int_0^{\frac{l}{2}} x\cos\frac{n\pi x}{l}\ dx\\
&= \frac{2(1+(-1)^n)}{n\pi}\int_0^{\frac{l}{2}} x\ d\left(\sin\frac{n\pi x}{l}\right)\\
&= \frac{2(1+(-1)^n)}{n\pi}\left(x\sin\frac{n\pi x}{l}\Big|_0^{\frac{l}{2}} - \int_0^{\frac{l}{2}}\sin\frac{n\pi x}{l}\ dx\right)\\
&= \frac{2(1+(-1)^n)}{n\pi}\left(\frac{l}{2}\sin\frac{n\pi}{2} + \frac{l}{n\pi}\int_0^{\frac{l}{2}}\ d\cos\frac{n\pi x}{l}\right)\\
&= \frac{2(1+(-1)^n)}{n\pi}\left[\frac{l}{2}\sin\frac{n\pi}{2} + \frac{l}{n\pi}\left(\cos\frac{n\pi}{2} - \cos 0\right)\right]\\
&= \begin{cases} 0, & \text{if } n \text{ is odd}\\ -\frac{4l}{n^2\pi^2}\cdot\left(1-(-1)^{\frac{n}{2}}\right), & \text{if } n \text{ is even.}\end{cases}\\
&= \begin{cases} -\frac{8l}{n^2\pi^2}, & \text{if } n = 2 \pmod 4;\ n = 4k-2,\ k = 1,\ 2,\ \ldots\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

It follows that

$$\begin{aligned}
f(x) &= \frac{l}{4} - \sum_{k=1}^{\infty}\frac{8l}{(4k-2)^2\pi^2}\cos\frac{(4k-2)\pi x}{l}\\
&= \frac{l}{4} - \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(4n-2)\pi x}{l}\ \text{ in } (0,l).
\end{aligned}$$

(c) Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ in $(0,l)$. Distinguish cases as above ($n = 0$ and $n \geq 1$):

$$\begin{aligned}
A_0 &= \frac{1}{l}\int_0^{\frac{l}{2}}\ dx\\
&= \frac{1}{2}
\end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} A_n &= \frac{2}{l}\int_0^{\frac{l}{2}} \cos\frac{n\pi x}{l}\, dx \\ &= \frac{2}{l}\cdot\frac{l}{n\pi}\sin x\Big|_0^{\frac{n\pi}{2}} = \frac{2}{n\pi}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n\pi}\cdot(-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

So,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{(2n-1)\pi x}{l}.$$

(d) Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ in $(0, l)$:

$$\begin{aligned} A_0 &= \frac{1}{l}\int_0^{l} \sin\frac{\pi x}{2l}\, dx \\ &= \frac{2}{\pi} \end{aligned}$$

For $n \geq 1$, we get

$$\begin{aligned} A_{n\geq 1} &= \frac{2}{l}\int_0^{l}\sin\frac{\pi x}{2l}\cos\frac{n\pi x}{l}\, dx \\ &= \frac{1}{l}\int_0^{l}\left(\sin\frac{(2n+1)\pi x}{2l} - \sin\frac{(2n-1)\pi x}{2l}\right) dx \\ &= \frac{1}{l}\left(\frac{2l}{(2n+1)\pi} - \frac{2l}{(2n-1)\pi}\right) \\ &= -\frac{4}{(4n^2-1)\pi} \end{aligned}$$

Then,

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\frac{n\pi x}{l} \qquad \text{in } (0, l).$$

(e) Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ in $(0, l)$. As I said in class, it is advisable that we distinguish the cases $n = 0$, $n = 1$ and $n \geq 2$ for the coefficients A_n. Accordingly,

$$\begin{aligned} A_0 &= \frac{1}{l}\int_0^{\frac{l}{2}}\sin\frac{\pi x}{l}\, dx \\ &= \frac{1}{\pi}, \end{aligned}$$

$$\begin{aligned} A_1 &= \frac{2}{l}\int_0^{\frac{l}{2}} \sin\frac{\pi x}{l}\cos\frac{\pi x}{l}\ dx \\ &= \frac{1}{l}\int_0^{\frac{l}{2}} \sin\frac{2\pi x}{l}\ dx \\ &= \frac{1}{\pi}, \end{aligned}$$

$$\begin{aligned} A_{n\geq 2} &= \frac{2}{l}\int_0^{\frac{l}{2}} \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\ dx \\ &= \frac{1}{l}\int_0^{\frac{l}{2}} \left(\sin\frac{(n+1)\pi x}{l} - \sin\frac{(n-1)\pi x}{l}\right)\ dx \\ &= \frac{1}{l}\left[\frac{l}{(n+1)\pi}\left(1-\cos\frac{n+1}{2}\pi\right) - \frac{l}{(n-1)\pi}\left(1-\cos\frac{n-1}{2}\pi\right)\right] \\ &= \frac{2n\sin\left(\frac{n\pi}{2}\right)-2}{(n^2-1)\pi}. \end{aligned}$$

It follows that

$$f(x) = \frac{1}{\pi}\left(1+\cos\frac{\pi x}{l} + 2\sum_{n=2}^{\infty}\frac{n\sin\left(\frac{n\pi}{2}\right)-1}{n^2-1}\cos\frac{n\pi x}{l}\right).$$

(f) Let $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$ in $(0,l)$:

$$\begin{aligned} A_0 &= \frac{1}{l}\int_0^{\epsilon}\frac{1}{\epsilon}\ dx \\ &= \frac{1}{l}, \\ A_{n\geq 1} &= \frac{2}{l}\int_0^{\epsilon}\frac{1}{\epsilon}\cos\frac{n\pi x}{l}\ dx \\ &= \frac{2}{\pi}\cdot\frac{\sin\left(\frac{n\pi\epsilon}{l}\right)}{n\epsilon}. \end{aligned}$$

It follows that

$$f(x) = \frac{1}{l} + \frac{2}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{\sin(\frac{n\pi\epsilon}{l})}{n}\cos\frac{n\pi x}{l}.$$