SOLUTION SET IV FOR 18.075–FALL 2004

10. Functions of a Complex Variable

10.12. **Residues.** .

In the following, I use the notation

$$Res_{z=z_0} f(z) \equiv Res(z_0) \equiv Res[f(z), z_0],$$

where Res is the residue of $f(z)$ at (the isolated singularity) z_0.

82. Evaluate the integral

$$\oint_C \frac{dz}{z^2-1}$$

when C is the curve sketched in Figure 10.21.

Solution. $\frac{1}{z^2-1}$ has two simple poles. One is at $z=1$, the other is at $z=-1$. It's easy to check that $Res[\frac{1}{z^2-1}, 1] = \frac{1}{2}$, and $Res[\frac{1}{z^2-1}, -1] = -\frac{1}{2}$. The pole at $z=1$ is encircled in the counterclockwise (positive) sense, while the pole at $z=-1$ is encircled in the clockwise sense. Hence,

$$\oint_C \frac{dz}{z^2-1} \;=\; 2\pi i Res[\frac{1}{z^2-1}, 1] - 2\pi i Res[\frac{1}{z^2-1}, -1] \;=\; \pi i - (-\pi i) \;=\; 2\pi i.$$

88 Determine the residue of each of the following functions at each singularity:
(a) $e^{\frac{1}{z}}$, (b) $e^{\frac{1}{z^2}}$, (c) $\cos\frac{\pi}{z-\pi}$, (d) $(1+z^2)e^{\frac{1}{z}}$.

Solution. (a) We have

$$e^{\frac{1}{z}} \;=\; \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \;=\; 1 + z^{-1} + \sum_{n=2}^{\infty} \frac{z^{-n}}{n!}.$$

So $e^{\frac{1}{z}}$ has an essential singularity at $z=0$, and

$$Res[e^{\frac{1}{z}}, 0] \;=\; 1.$$

(b) We have

$$e^{\frac{1}{z^2}} \;=\; \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} \;=\; 1 + z^{-2} + \sum_{n=2}^{\infty} \frac{z^{-2n}}{n!}.$$

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So $e^{\frac{1}{z^2}}$ has an essential singularity at $z = 0$, and

$$Res[e^{\frac{1}{z^2}}, 0] = 0.$$

(c) We have

$$\cos\frac{\pi}{z-\pi} = \sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(z-\pi)^{2n}(2n)!}.$$

(Note that the coefficient of $\frac{1}{z-\pi}$ is 0.) So $\cos\frac{\pi}{z-\pi}$ has an essential singularity at $z = \pi$, and

$$Res[\cos\frac{\pi}{z-\pi}, \pi] = 0.$$

(d) We have

$$\begin{aligned}(1+z^2)e^{\frac{1}{z}} &= (1+z^2)\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} \\ &= (1+z^2)(1+z^{-1}+\frac{z^{-2}}{2}+\frac{z^{-3}}{6}+\sum_{n=4}^{\infty}\frac{z^{-n}}{n!}) \\ &= \frac{3}{2}+\frac{7}{6}z^{-1}+[\textit{higher powers of } z^{-1}].\end{aligned}$$

So $(1+z^2)e^{\frac{1}{z}}$ has an essential singularity at $z = 0$, and

$$Res[(1+z^2)e^{\frac{1}{z}}, 0] = \frac{7}{6}.$$

10.13. **10.13 Evaluation of Real Definite Integrals.** .

90. Use residue calculus to evaluate the following integrals:
(a) $\int_0^{2\pi}\frac{d\theta}{A+B\sin\theta} = \frac{2\pi}{\sqrt{A^2-B^2}}$ $(A > |B|)$,
(b) $\int_0^{2\pi}\frac{d\theta}{a^2+\sin^2\theta} = \int_0^{2\pi}\frac{d\theta}{a^2+\cos^2\theta} = \frac{2\pi}{a\sqrt{a^2+1}}$ $(a > 0)$,
(c) $\int_0^{\frac{\pi}{2}}\sin^4\theta\, d\theta = \int_0^{\frac{\pi}{2}}\cos^4\theta\, d\theta = \frac{3\pi}{16}$,
(d) $\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{4}$.

Solution. (a) First make the substitution: $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$.

Now the complex z describes the unit circle C_1 in the positive sense as θ varies from 0 to 2π. So, as was discussed in class, the integral becomes

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{A+B\sin\theta} &= \oint_{C_1}\frac{\frac{dz}{iz}}{A+B\frac{z^2-1}{2iz}} \\ &= \oint_{C_1}\frac{2dz}{Bz^2+2iAz-B}\end{aligned}$$

The poles of the integrand are simple and occur when $Bz^2 + 2iA\,z - B = 0$, which in turn gives

$$z_\pm = \frac{-iA \pm \sqrt{B^2 - A^2}}{B}$$

Furthermore,

$$z_+ z_- = -\frac{A^2}{B^2} + \frac{A^2 - B^2}{B^2} = -1.$$

Therefore, z_+ is a (simple) pole inside the unit circle. Now using the known formula $Res_{z=z_0} \frac{g(z)}{h(z)} = \frac{g(z_0)}{h'(z_0)}$, where z_0: simple zero of $h(z)$, we get:

$$\begin{aligned} Res(z_+) &= \frac{2}{2Bz_+ + 2iA} \\ &= \frac{1}{-iA + i\sqrt{A^2 - B^2} + iA} \\ &= \frac{1}{i\sqrt{A^2 - B^2}} \end{aligned}$$

Thus, by the residue theorem,

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{A + B\sin\theta} &= \frac{2\pi i}{i\sqrt{A^2 - B^2}} \\ &= \frac{2\pi}{\sqrt{A^2 - B^2}} \end{aligned}$$

(b) We manipulate the integrand as follows:

$$\frac{1}{a^2 + \sin^2\theta} = \frac{1}{a^2 + \frac{1-\cos 2\theta}{2}} = \frac{2}{2a^2 + 1 - \cos 2\theta}.$$

So, with the new variable $\varphi = 2\theta$,

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a^2 + \sin^2\theta} &= \int_0^{2\pi} \frac{2d\theta}{2a^2 + 1 - \cos 2\theta} \\ &= \int_0^{4\pi} \frac{d\varphi}{2a^2 + 1 - \cos\varphi} \\ &= 2\int_0^{2\pi} \frac{d\varphi}{2a^2 + 1 - \cos\varphi} \\ &= 2\int_0^{2\pi} \frac{d\varphi}{2a^2 + 1 - \sin\varphi}, \end{aligned}$$

where we shifted the integration variable by $\pi/2$ in the integral of the third line and used the periodicity of the integrand.

By using the result of part (a) above with $A = 2a^2 + 1$ and $B = -1$, we get

$$\int_0^{2\pi} \frac{d\varphi}{2a^2 + 1 - \sin\varphi} = \frac{2\pi}{\sqrt{(2a^2+1)^2 - 1}} = \frac{\pi}{a\sqrt{a^2+1}}.$$

Thus,

$$\int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta} = \frac{2\pi}{a\sqrt{a^2+1}},$$

and, hence,

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a^2+\cos^2\theta} &= \int_0^{2\pi} \frac{d\theta}{a^2+\sin^2(\theta-\frac{\pi}{2})} \\ &= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\theta}{a^2+\sin^2\theta} \\ &= \int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta} \\ &= \frac{\pi}{a\sqrt{a^2+1}}.\end{aligned}$$

The integral of the third line ensues from the periodicity of the integrand.

(c) By subtracting the given integrals, we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^4\theta\ d\theta - \int_0^{\frac{\pi}{2}} \cos^4\theta\ d\theta &= \int_0^{\frac{\pi}{2}} (\sin^4\theta-\cos^4\theta)\ d\theta \\ &= \int_0^{\frac{\pi}{2}} (\sin^2\theta-\cos^2\theta)(\sin^2\theta+\cos^2\theta)\ d\theta \\ &= \int_0^{\frac{\pi}{2}} (\sin^2\theta-\cos^2\theta)\ d\theta \\ &= -\int_0^{\frac{\pi}{2}} \cos 2\theta\ d\theta \\ &= -\frac{\sin 2\theta}{2}\Big|_0^{\frac{\pi}{2}} \\ &= 0.\end{aligned}$$

So,

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\ d\theta = \int_0^{\frac{\pi}{2}} \cos^4\theta\ d\theta.$$

Then,

$$\begin{aligned}\int_0^{2\pi} \sin^4\theta\ d\theta &= \int_0^{\frac{\pi}{2}} \sin^4\theta\ d\theta + \int_{\frac{\pi}{2}}^{\pi} \sin^4\theta\ d\theta + \int_{\pi}^{\frac{3\pi}{2}} \sin^4\theta\ d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \sin^4\theta\ d\theta \\ &= \int_0^{\frac{\pi}{2}} [\sin^4\theta + \sin^4(\theta+\frac{\pi}{2}) + \sin^4(\theta+\pi) + \sin^4(\theta+\frac{3\pi}{2})]\ d\theta \\ &= 2\int_0^{\frac{\pi}{2}} (\sin^4\theta+\cos^4\theta)\ d\theta \\ &= 4\int_0^{\frac{\pi}{2}} \sin^4\theta\ d\theta.\end{aligned}$$

This shows

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\ d\theta \ = \ \int_0^{\frac{\pi}{2}} \cos^4\theta\ d\theta \ = \ \frac{1}{4}\int_0^{2\pi} \sin^4\theta\ d\theta.$$

But

$$\begin{aligned}\int_0^{2\pi} \sin^4\theta\ d\theta \ &= \ \oint_{C_1} (\frac{z^2-1}{2iz})^4\ \frac{dz}{iz} \\ &= \ \frac{1}{16i}\oint_{C_1} \frac{(z^2-1)^4}{z^5}\ dz \\ &= \ \frac{2\pi i}{16i}\ Res[\frac{(z^2-1)^4}{z^5}, 0] \\ &= \ \frac{2\pi i}{16i}\ 6 \\ &= \ \frac{3\pi}{4},\end{aligned}$$

where C_1 is the unit circle with center at origin. The residue was found easily by noticing that

$$\frac{(z^2-1)^4}{z^5} = \frac{z^8-4z^6+6z^4-4z^2+1}{z^5},$$

by which the coefficient of z^{-1} is 6. So,

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\ d\theta \ = \ \int_0^{\frac{\pi}{2}} \cos^4\theta\ d\theta \ = \ \frac{3\pi}{16}.$$

(d) Again, by the usual replacement $z = e^{i\theta}$,

$$\begin{aligned}\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta \ &= \ \oint_{C_1} \frac{\frac{(z^2-1)^2}{(2iz)^2}}{5+4\frac{z^2+1}{2z}}\frac{dz}{iz} \\ &= \ \oint_{C_1} \frac{(z^2-1)^2}{-2iz^2(10z+4z^2+4)}\ dz\end{aligned}$$

The simple poles of this integrand occur when $4z^2+10z+4=0$, i.e., when $z = -\frac{1}{2}$ or $z = 2$, while a double pole occurs at $z = 0$. Since $z = 2$ is not within the unit circle, we disregard it.

$$
\begin{aligned}
Res(-\frac{1}{2}) &= \lim_{z\to-\frac{1}{2}} \frac{(z+\frac{1}{2})\cdot(z^2-1)^2}{-2iz^2(2z+1)(z+2)} \\
&= \frac{(\frac{-3}{4})^2}{-2i(\frac{3}{2})} \\
&= \frac{3i}{16} \\
Res(0) &= \frac{1}{(2-1)!}\left[\frac{d}{dz}(z^2 f(z))\right]_{z=0} \\
&= \left[\frac{d}{dz}\left(\frac{(z^2-1)^2}{-2i(4z^2+10z+4)}\right)\right]_{z=0} \\
&= \left[\frac{-4i(2z^2+5z+2)(4z^3-4z)-(z^4-2z^2+1)(-16iz-20i)}{(-4i)^2(2z^2+5z+2)^2}\right]_{z=0} \\
&= \frac{-5i}{16}
\end{aligned}
$$

Alternatively, you may expand the integrand in z (considering $|z|$ "small") and find the coefficient of z^{-1}. (Try it for practice!)

Thus,

$$
\begin{aligned}
\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta &= 2\pi i\left(\frac{3i}{16}-\frac{5i}{16}\right) \\
&= \frac{\pi}{4}
\end{aligned}
$$

91. Use residue calculus to evaluate the following integrals:
(a) $\int_{-\infty}^{\infty} \frac{dx}{(x+b)^2+a^2} = \frac{\pi}{a}$ $(a>0)$,
(b) $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$ $(a>0,\ b>0)$,
(c) $\int_0^{\infty} \frac{dx}{x^4+4a^4} = \frac{\pi}{8a^3}$ $(a>0)$,
(d) $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$ $(a>0)$.

Solution.

(a) The degree of the denominator is 2 greater than the degree of the numerator and the function is finite for all real values of x. Thus, we can employ the strategy given in class by closing the original path with a large semicircle in the upper half plane (or lower half plane). By shifting the integration variable by $-b$, we get

$$
\int_{-\infty}^{\infty} \frac{dx}{(x+b)^2+a^2} = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \oint_{C_1} F(z)dz = 2\pi i\sum_k Res(z_k)
$$

where the points z_k are the poles of $F(z) = \frac{1}{z^2+a^2}$ in the upper half-plane.

The (simple) poles occur when $z^2 + a^2 = 0$, that is when $z = \pm i\sqrt(a^2) = \pm ia$ (since $a > 0$). So there is one (simple) pole in the upper half-plane, namely, at $z_1 = ia$.

$$Res(z_1) = \frac{1}{2z_1} = \frac{1}{2ia}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{dx}{(x+b)^2 + a^2} = 2\pi i \cdot \frac{1}{2ia} = \frac{\pi}{a}.$$

(b) $\frac{1}{(z^2+a^2)(z^2+b^2)}$ has two singularities on the upper half plane. One of these is at $z = ai$, the other is at $z = bi$; both of them are simple poles. Note that the denominator of $\frac{1}{(x^2+a^2)(x^2+b^2)}$ is of degree 4. Accordingly,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} &= 2\pi i \ (Res[\frac{1}{(z^2+a^2)(z^2+b^2)}, ai] + Res[\frac{1}{(z^2+a^2)(z^2+b^2)}, bi]) \\
&= 2\pi i \ (\frac{1}{2ai(b^2-a^2)} + \frac{1}{2bi(a^2-b^2)}) \\
&= \frac{\pi}{ab(a+b)}.
\end{aligned}$$

Since $\frac{1}{(x^2+a^2)(x^2+b^2)}$ is even, we get

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}.$$

(c) $\frac{1}{z^4+4a^4}$ has two singularities on the upper half plane. One of these is at $z = \sqrt{2}e^{\frac{\pi i}{4}}a$, the other is at $z = \sqrt{2}e^{\frac{3\pi i}{4}}a$. Both of them are simple poles. Note that the degree of the denominator of $\frac{1}{x^4+4a^4}$ is 4. Accordingly,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{x^4+4a^4} &= 2\pi i \ (Res[\frac{1}{z^4+4a^4}, \sqrt{2}e^{\frac{\pi i}{4}}a] + Res[\frac{1}{z^4+4a^4}, \sqrt{2}e^{\frac{3\pi i}{4}}a]) \\
&= 2\pi i \ (\frac{e^{-\frac{\pi i}{4}}}{8\sqrt{2}a^3 i} - \frac{e^{-\frac{3\pi i}{4}}}{8\sqrt{2}a^3 i}) \\
&= \frac{\pi}{4\sqrt{2}a^3} \ [(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) - (-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i)] \\
&= \frac{\pi}{4a^3}.
\end{aligned}$$

Since $\frac{1}{x^4+4a^4}$ is even, we have

$$\int_{0}^{\infty} \frac{dx}{x^4+4a^4} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4+4a^4} = \frac{\pi}{8a^3}.$$

(d) The degree of the denominator is greater than twice the degree of the numerator and the function is finite for all real values of x. We can once again employ the strategy given in class. Also, note that the integrand is even so that

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

The poles occur when $(z^2+a^2)^2 = 0$, that is when $z = \pm ia$. Thus, there is one pole in the upper half-plane, i.e., at $z = ia$, and it is a double pole.

$$\begin{aligned} Res(ia) &= \left[\frac{d}{dz}\frac{(z-ia)^2}{(z^2+a^2)^2}\right]_{z=ia} \\ &= \left[\frac{d}{dz}\frac{1}{(z+ia)^2}\right]_{z=ia} \\ &= \left[\frac{-2}{(z+ia)^3}\right]_{z=ia} \\ &= \frac{1}{4ia^3} \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty \frac{dx}{(x^2+a^2)^2} &= \frac{1}{2}\cdot 2\pi i \frac{1}{4ia^3} \\ &= \frac{\pi}{4a^3} \end{aligned}$$

92. Use residue calculus to evaluate the following integrals:
(a) $\int_0^\infty \frac{x \sin mx}{a^2+x^2}\, dx = \frac{\pi}{2}\, e^{-am}$ $(a>0,\, m>0)$,
(b) $\int_0^\infty \frac{\cos mx}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{2(b^2-a^2)}\, (\frac{e^{-am}}{a} - \frac{e^{-bm}}{b})$ $(a>0,\, b>0,\, m\geq 0,\, b\neq a)$,
(c) $\int_{-\infty}^\infty \frac{\cos mx}{(x+b)^2+a^2}\, dx = \frac{\pi}{a}\, e^{-am}\cos bm$ $(a>0,\, m\geq 0)$,
$\int_{-\infty}^\infty \frac{\sin mx}{(x+b)^2+a^2}\, dx = -\frac{\pi}{a}\, e^{-am}\sin bm$ $(a>0,\, m\geq 0)$,
(d) $\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3} e^{-am}(1+am)$ $(a>0,\, m\geq 0)$,
(e) $\int_0^\infty \frac{\cos mx}{x^4+4a^4}\, dx = \frac{\pi}{8a^3}\, e^{-am}(\cos am + \sin am)$ $(a>0,\ m\geq 0)$,
(f) $\int_0^\infty \frac{x^3 \sin mx}{x^4+4a^4} dx = \frac{\pi}{2} e^{-am}\cos am$ $(a>0,\, m>0)$.

Solution. (a) $\frac{z}{a^2+z^2}$ has a simple pole in the upper half plane, which is at $z = ai$.

$$\begin{aligned} \int_{-\infty}^\infty \frac{x\cos mx}{a^2+x^2}\, dx + i\int_{-\infty}^\infty \frac{x \sin mx}{a^2+x^2}\, dx &= \int_{-\infty}^\infty \frac{xe^{mxi}}{a^2+x^2}\, dx \\ &= 2\pi i\ Res[\frac{ze^{mzi}}{a^2+z^2}, ai] \\ &= 2\pi i\ \frac{aie^{-am}}{2ai} \\ &= \pi i\ e^{-am}, \end{aligned}$$

where we close the path in the upper half plane for the last integral involving e^{imz}, since $m>0$. The first integral in the first line is of course 0 because the integrand is odd. So,

$$\int_{-\infty}^\infty \frac{x\sin mx}{a^2+x^2}\, dx = \pi e^{-am}.$$

Note that $\frac{x \sin mx}{a^2+x^2}$ is even, we have

$$\int_0^\infty \frac{x \sin mx}{a^2+x^2}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x \sin mx}{a^2+x^2}\, dx = \frac{\pi}{2}\, e^{-am}.$$

(b) $\frac{1}{z^2+a^2}$ has a simple pole on the upper half plane, which it at $z = ai$. Since $m \geq 0$, we get

$$\begin{aligned}\int_{-\infty}^\infty \frac{\cos mx}{x^2+a^2}\, dx + i\int_{-\infty}^\infty \frac{\sin mx}{x^2+a^2}\, dx &= \int_{-\infty}^\infty \frac{e^{mxi}}{x^2+a^2}\, dx \\ &= 2\pi i\ Res[\frac{e^{mxi}}{z^2+a^2}, ai] \\ &= 2\pi i\ \frac{e^{-am}}{2ai} \\ &= \frac{\pi e^{-am}}{a},\end{aligned}$$

where we close the path in the upper half plane. The second integral in the first line vanishes because the integrand is odd. So,

$$\int_{-\infty}^\infty \frac{\cos mx}{x^2+a^2}\, dx = \frac{\pi e^{-am}}{a}.$$

Since $\frac{\cos mx}{x^2+a^2}$ is even, we have

$$\int_0^\infty \frac{\cos mx}{x^2+a^2}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos mx}{x^2+a^2}\, dx = \frac{\pi e^{-am}}{2a}.$$

Similarly,

$$\int_0^\infty \frac{\cos mx}{x^2+b^2}\, dx = \frac{\pi e^{-bm}}{2b}.$$

So,

$$\begin{aligned}\int_0^\infty \frac{\cos mx}{(x^2+a^2)(x^2+b^2)}\, dx &= \int_0^\infty \frac{1}{b^2-a^2}\,(\frac{\cos mx}{x^2+a^2} - \frac{\cos mx}{x^2+b^2})\, dx \\ &= \frac{1}{b^2-a^2}\,(\int_0^\infty \frac{\cos mx}{x^2+a^2}\, dx - \int_0^\infty \frac{\cos mx}{x^2+b^2}\, dx) \\ &= \frac{1}{b^2-a^2}\,(\frac{\pi e^{-am}}{2a} - \frac{\pi e^{-bm}}{2b}) \\ &= \frac{\pi}{2(b^2-a^2)}\,(\frac{e^{-am}}{a} - \frac{e^{-bm}}{b}).\end{aligned}$$

(c) The given integrals are evaluated by the standard prescription as follows.

$$\int_{-\infty}^\infty \frac{\cos mx}{(x+b)^2+a^2}\, dx = Re\int_{-\infty}^\infty \frac{e^{imx}}{(x+b)^2+a^2}\, dx = Re\int_{-\infty}^\infty \frac{e^{im(x-b)}}{x^2+a^2}\, dx$$

$$\int_{-\infty}^\infty \frac{\sin mx}{(x+b)^2+a^2}\, dx = Im\int_{-\infty}^\infty \frac{e^{imx}}{(x+b)^2+a^2}\, dx = Im\int_{-\infty}^\infty \frac{e^{im(x-b)}}{x^2+a^2}\, dx,$$

where

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+a^2}\,dx = 2\pi i\, Res_{z=ia} \frac{e^{imz}}{z^2+a^2} = \frac{\pi}{a}\, e^{-ma}.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x+b)^2+a^2}\,dx = Re\left(e^{-imb}\,\frac{\pi}{a}\,e^{-ma}\right) = \frac{\pi}{a}\,e^{-ma}\,\cos mb,$$

$$\int_{-\infty}^{\infty} \frac{\sin mx}{(x+b)^2+a^2}\,dx = Im\left(e^{-imb}\,\frac{\pi}{a}\,e^{-ma}\right) = -\frac{\pi}{a}\,e^{-ma}\,\sin mb.$$

(d) We can calculate the real part of $\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}dx$ by noticing that our function is even. As in question 91(d), we have one pole in the upper half-plane. This is a double pole at $z = ia$.

$$\begin{aligned}
Res(ia) &= \left[\frac{d}{dz}\left(\frac{(z-ia)^2 e^{imz}}{(z^2+a^2)^2}\right)\right]_{z=ia} \\
&= \left[\frac{d}{dz}\left(\frac{e^{imz}}{(z+ia)^2}\right)\right]_{z=ia} \\
&= \left[\frac{(z+ia)^2 ime^{imz} - e^{imz}2(z+ia)}{(z+ia)^4}\right]_{z=ia} \\
&= \frac{(2ia)^2 ime^{-ma} - e^{-ma}4ia}{(2ia)^4} \\
&= -ie^{-ma}\left(\frac{ma+1}{4a^3}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{\infty} \frac{\cos mx}{(x^2+a^2)^2}dx &= \frac{1}{2}Re\left[\int_{-\infty}^{\infty} \frac{e^{imx}}{(x^2+a^2)^2}dx\right] \\
&= \frac{1}{2}Re\left[2\pi i \cdot -ie^{-ma}\left(\frac{ma+1}{4a^3}\right)\right] \\
&= \pi e^{-ma}\left(\frac{ma+1}{4a^3}\right)
\end{aligned}$$

(e) Clearly,

$$\int_0^{\infty} \frac{\cos mx}{x^4+4a^4}\,dx = \frac{1}{2}\,Re\int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+4a^4}\,dx.$$

We calculate the last integral involving e^{imz} by closing the path in the upper half plane since $m > 0$. The poles of the integrand occur at $z^4 + 4a^4 = 0$ and are all simple. The poles that lie in the upper half plane are $z_1 = e^{i\pi/4}\sqrt{2}a$ and $z_2 = e^{3i\pi/4}\sqrt{2}a$. Accordingly,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+4a^4}\,dx &= 2\pi i \left(Res_{z=z_1} \frac{e^{imz}}{z^4+4a^4} + Res_{z=z_2} \frac{e^{imz}}{z^4+4a^4} \right) \\
&= 2\pi i \left(\frac{e^{imz_1}}{4z_1^3} + \frac{e^{imz_2}}{4z_2^3} \right) \\
&= \frac{2\pi i}{16a^3}[-(cosma + i\sin ma)(1+i) + (\cos ma - i\sin ma)(1-i)] \\
&= \frac{\pi}{4a^3}(\cos ma + \sin ma)
\end{aligned}$$

It follows that

$$\int_0^{\infty} \frac{\cos mx}{x^4+4a^4}\,dx = \frac{\pi}{8a^3}(\cos ma + \sin ma).$$

(f) Once again, we can calculate the imaginary part of $\int_0^{\infty} \frac{x^3 e^{imx}}{x^4+4a^4} dx$ noting that our function is even and $m > 0$. The poles occur when $z^4 + 4a^4 = 0 \Rightarrow z = \pm\sqrt{\pm 2ia^2}$, as in part (e) above. Thus the roots are:

$$\begin{aligned}
a_1 &= \sqrt{2ia^2} = a\sqrt{2}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = a + ia \text{ since } a > 0. \\
a_2 &= -a_1 = -a - ia. \\
a_3 &= \sqrt{-2ia^2} = ia - a. \\
a_4 &= -a_3 = -ia + a.
\end{aligned}$$

Thus there are two poles in the upper half-plane: a_1 and a_3 (both simple poles).

$$\begin{aligned}
Res(a_1) &= \lim_{z\to a_1} (z - a_1) f(z) \\
&= \lim_{z\to a_1} \frac{(z - a - ia) z^3 e^{imz}}{z^4 + 4a^2} \\
&= \lim_{z\to a_1} \frac{z^3 e^{imz}}{(z + a + ia)(z^2 + 2ia^2)} \\
&= \frac{(a + ia)^2 e^{im(a+ia)}}{8ia^2}
\end{aligned}$$

$$\begin{aligned}
Res(a_3) &= \lim_{z\to a_3} \frac{z^3 e^{imz}}{(z + ia - a)(z^2 - 2ia^2)} \\
&= \frac{(ia - a)^2 e^{-ima - ma}}{-8ia}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^\infty \frac{x^3 \sin mx}{x^4+4a^4}dx &= \frac{1}{2} Im\left[\int_{-\infty}^\infty \frac{x^3 e^{imx}}{x^4+4a^4}dx\right] \\
&= \frac{1}{2} Im\left[2\pi\left(\frac{(a+ia)^2 e^{ima-ma} - (ia-a)^2 e^{-ima-ma}}{8a}\right)\right] \\
&= \frac{1}{2} Im\left[\frac{\pi}{2} i(e^{ima}\cdot e^{-ma} + e^{-ima}\cdot e^{-ma})\right] \\
&= \frac{\pi}{2} e^{-ma}\left(\frac{e^{ima}+e^{-ima}}{2}\right) \\
&= \frac{\pi}{2} e^{-ma} \cos ma
\end{aligned}$$