Solutions to 18.075 In-Class Practice Test I for Exam 2

(I) $I = \int_0^\infty dx \, \frac{\cos x}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} dx \, \frac{\cos x}{1+x^2} = \frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{1+x^2}$

$\frac{e^{iz}}{1+z^2}$ has simple poles at $1+z^2=0 \Leftrightarrow z = \pm i$

We close the path by a large semicircle C_R of radius R in the upper half plane and allow $R \to +\infty$. Define $C = C_R + C_1(R)$ where $C_1(R) = (-R, R)$.

By the residue theorem,

- $$\oint_C dz \, \frac{e^{iz}}{1+z^2} = 2\pi i \operatorname*{Res}_{z=i} \left(\frac{e^{iz}}{z^2+1} \right) = 2\pi i \, \frac{e^{i \cdot i}}{2i} = \pi e^{-1}$$

- $\oint_C = \int_{C_R} + \int_{C_1}$. By Theorem 2 given in class about limiting contours,

$$\lim_{R\to\infty} \int_{C_R} dz \, \frac{e^{iz}}{1+z^2} = 0.$$

Hence, in the limit $R \to \infty$,

$$\frac{1}{2} \int_{C_1} dz \, \frac{e^{iz}}{1+z^2} = \frac{1}{2} \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{1+x^2} = \frac{1}{2} \oint dz \, \frac{e^{iz}}{1+z^2} = \frac{\pi}{2} e^{-1}$$

So, $I = \frac{\pi}{2} e^{-1}$.

(II) $$I = \int_{-\infty}^{\infty} \frac{\cos x}{(4x^2-\pi^2)(x^2+4)}\,dx$$

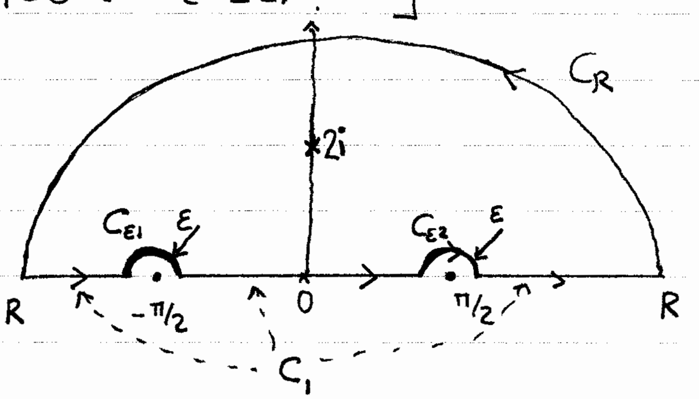
(1) The integrand, $\dfrac{\cos z}{(4z^2-\pi^2)(z^2+4)}$, has simple poles only at $z^2+4=0 \Leftrightarrow z=\pm 2i$

[By writing

$$I = \overset{\text{principal value}}{P}\int_{-\infty}^{\infty} \frac{\cos x}{(4x^2-\pi^2)(x^2+4)}\,dx = \mathrm{Re}\,P\int_{-\infty}^{\infty} \frac{e^{ix}}{(4x^2-\pi^2)(x^2+4)}\,dx,$$

the new integrand, $\dfrac{e^{iz}}{(4z^2-\pi^2)(z^2+4)}$, has simple poles at

$4z^2-\pi^2=0 \Leftrightarrow z=\pm\pi/2$ <u>and</u> at $z^2+4=0 \Leftrightarrow z=\pm 2i$.]



(2) $$I = \mathrm{Re}\,P\int_{-\infty}^{\infty} \frac{e^{ix}}{(4x^2-\pi^2)(x^2+4)}\,dx$$

$$= \lim_{R\to\infty} \mathrm{Re}\int_{C_1} \frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)}\,dz,$$

where $C_1 = (-R, -\pi/2-\epsilon) + (-\pi/2+\epsilon, \pi/2-\epsilon) + (\pi/2+\epsilon, R)$ as $R\to\infty$.

Let $C = C_1 + C_{\varepsilon_1} + C_{\varepsilon_2} + C_R$. By Theorem 2, $\displaystyle\lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)}\,dz = 0.$

(3) By residue theorem, $$\oint_C dz\,\frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} = 2\pi i \operatorname*{Res}_{z=2i}\left[\frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)}\right]$$

$$= 2\pi i\,\frac{e^{2i\cdot i}}{4\cdot 4\cdot i^2-\pi^2}\,\frac{1}{2\cdot 2i} = -\frac{1}{2}\,\frac{\pi e^{-2}}{\pi^2+16}.$$

$$\oint_C = \int_{C_1} + \int_{C_{\varepsilon_1}} + \int_{C_{\varepsilon_2}} + \int_{C_R}.$$

By Theorem 2, $\displaystyle \lim_{R\to\infty} \int_{C_R} dz \, \frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} = 0.$

By Theorem 4,

$$\lim_{\varepsilon\to 0^+} \int_{C_{\varepsilon_1}} dz \, \frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} = -i\pi \cdot \operatorname*{Res}_{z=-\frac{\pi}{2}} \left(\frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} \right)$$

$$= -\frac{i\pi}{4} \operatorname*{Res}_{z=-\frac{\pi}{2}} \left[\frac{e^{iz}}{(z^2-\pi^2/4)(z^2+4)} \right]$$

$$= -\frac{i\pi}{4} \frac{e^{-i\pi/2}}{(-\cancel{2}\frac{\pi}{\cancel{2}})(\frac{\pi^2}{4}+4)} = \frac{1}{\pi^2+16},$$

$$\lim_{\varepsilon\to 0^+} \int_{C_{\varepsilon_2}} dz \, \frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} = -i\pi \operatorname*{Res}_{z=\pi/2} \left[\frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} \right]$$

$$= -\frac{i\pi}{4} \frac{e^{i\pi/2}}{(\frac{\pi^2}{4}+4)\,\cancel{2}\frac{\pi}{\cancel{2}}} = \frac{1}{\pi^2+16}.$$

By putting all pieces together we get:

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left[\int_{C_1(R)} + \int_{C_{\varepsilon_1}} + \int_{C_{\varepsilon_2}} + \cancelto{0}{\int_{C_R}} \right] dz \, \frac{e^{iz}}{(4z^2-\pi^2)(z^2+4)} = -\frac{1}{2} \frac{\pi e^{-2}}{\pi^2+16}$$

$$\Leftrightarrow \quad I = -\frac{1}{2} \frac{\pi e^{-2}}{\pi^2+16} - \frac{2}{\pi^2+16} = -\left(\frac{\pi}{2} e^{-2} + 2 \right) \frac{1}{\pi^2+16}.$$

III $$I = \int_0^{\pi} \frac{\sin^2\theta}{2+\cos^2\theta}\, d\theta$$

1. Try to simplify this integral a bit:

$$\sin^2\theta = 1-\cos^2\theta = 3-(2+\cos^2\theta)$$

$$I = \int_0^{\pi} \frac{3-(2+\cos^2\theta)}{2+\cos^2\theta}\, d\theta = 3\int_0^{\pi} \frac{d\theta}{2+\cos^2\theta} - \pi .$$

Use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$I = 3\int_0^{\pi} \frac{d\theta}{2+\frac{1+\cos 2\theta}{2}} - \pi = \cancel{3}\int_0^{\pi} \frac{2d\theta}{5+\cos 2\theta} - \pi \overset{\varphi=2\theta}{=\!=} 3\int_0^{2\pi} \frac{d\varphi}{5+\cos\varphi} - \pi .$$

Let $z = e^{i\varphi}$, $d\varphi = \frac{dz}{iz}$, $\cos\varphi = \frac{z+z^{-1}}{2}$.

$$I = 3\oint_C \frac{dz}{iz}\, \frac{1}{5+\frac{z+z^{-1}}{2}} - \pi \quad , \quad C: \text{unit circle}$$

$$= 3\oint_C \frac{dz}{iz}\, \frac{1}{5+\frac{1}{2z}(z^2+1)} - \pi = 3\oint_C \frac{dz}{i\cancel{z}}\, \frac{2\cancel{z}}{z^2+1+10z} - \pi$$

$$= \frac{6}{i}\oint_C dz\, \frac{1}{z^2+10z+1} - \pi$$

2. Poles occur at $z^2+10z+1=0 \Leftrightarrow z_{\pm} = -5 \pm \sqrt{24} = -5 \pm 2\sqrt{6}$

z_+ only is inside the unit circle.

Residue Theorem:

$$I = \frac{6}{i}\, 2\pi i \operatorname*{Res}_{z=z_+}\left(\frac{1}{z^2+10z+1}\right) - \pi = \frac{6}{i}\, 2\pi i\, \frac{1}{2z_+ + 10} - \pi = \frac{6}{i}\, \frac{2\pi i}{\cancel{-10}+4\sqrt{6}+\cancel{10}} - \pi$$

$$= \frac{6}{i}\, \frac{2\pi i}{4\sqrt{6}} - \pi = \frac{3\pi}{\sqrt{6}} - \pi = \frac{\pi\sqrt{6}}{2} - \pi = \left(\frac{\sqrt{6}}{2} - 1\right)\pi$$

(IV). ① $\displaystyle\sum_{n=0}^{\infty} a_n (x+1)^n, \quad a_n = \frac{(-2)^n}{n!}$

Ratio test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n \frac{2^n}{n!}}{(-1)^{n+1}\frac{2^{n+1}}{(n+1)!}}\right| = \lim_{n\to\infty}\left[(n+1)\frac{1}{2}\right] = \infty.$

So, $R = \infty$.

② $\displaystyle\sum_{n=0}^{\infty} a_n x^n, \quad a_n = \begin{cases} n(n+1), & n: \text{even} \\ n^2, & n: \text{odd.} \end{cases}$

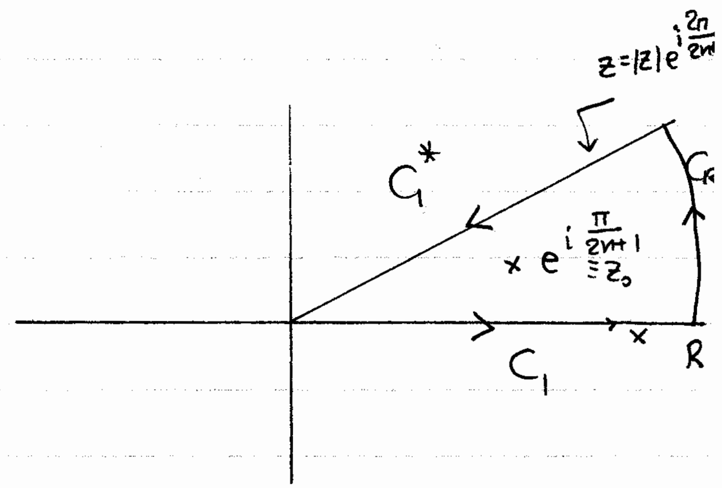
Ratio test:

$\underline{n: \text{odd}}$: $\displaystyle\left|\frac{a_n}{a_{n+1}}\right| = \frac{n^2}{(n+1)(n+2)} \xrightarrow[n\to\infty]{} 1$

$\underline{n: \text{even}}$: $\displaystyle\left|\frac{a_n}{a_{n+1}}\right| = \frac{n(n+1)}{(n+1)^2} = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1.$

So, the limit exists. $R = 1$.

(V) $\displaystyle I = \int_0^{\infty} \frac{x}{1+x^{2n+1}}\, dx$



The integrand has simple poles at

$$1+z^{2n+1} = 0 \iff z = z_k = e^{(i\pi + 2ik\pi)\frac{1}{2n+1}},$$
$$k = 0, 1, \ldots, 2n.$$

Let $C = C_1 + C_R + C_1^*$, where C_1^* is the ray with $z = |z| e^{i\frac{2\pi}{2n+1}}$.

Then, as $R\to\infty$, $\displaystyle\int_{C_1^*} \frac{z}{1+z^{2n+1}}\, dz = -e^{i\frac{4\pi}{2n+1}} I,$

because, with $|z| = x$, $dz = e^{i\frac{2\pi}{2n+1}} dx$,
$z = e^{i\frac{2\pi}{2n+1}} x$.

On the other hand,

$$\oint_C dz \, \frac{z}{1+z^{2n+1}} = 2\pi i \cdot \operatorname*{Res}_{z=z_0} \left[\frac{z}{1+z^{2n+1}} \right]$$

and

$$\lim_{R\to\infty} \int_{C_R} dz \, \frac{z}{1+z^{2n+1}} = 0 \quad \text{by Theorem 1.}$$

Hence,

$$\left(1 - e^{i\frac{4\pi}{2n+1}}\right) I = 2\pi i \operatorname*{Res}_{z=e^{i\frac{\pi}{2n+1}}} \left[\frac{z}{1+z^{2n+1}} \right] = 2\pi i \, \frac{e^{i\frac{\pi}{2n+1}}}{(2n+1)\, e^{i\frac{2n\pi}{2n+1}}}$$

$$\Leftrightarrow \; -e^{i\frac{2\pi}{2n+1}} \cancel{2i} \sin\left(\frac{2\pi}{2n+1}\right) I = \cancel{2}\pi \cancel{i} \, \frac{e^{i\frac{\pi}{2n+1}}}{(2n+1)\, e^{i\frac{2n\pi}{2n+1}}}$$

$$\Leftrightarrow \; -I \sin\left(\frac{2\pi}{2n+1}\right) = \frac{\pi}{2n+1} \underbrace{\left(\frac{e^{-i\frac{\pi}{2n+1}}}{e^{i\frac{2n\pi}{2n+1}}} \right)}_{=-1} \;\Leftrightarrow\; I = \frac{\pi/(2n+1)}{\sin\left(\frac{2\pi}{2n+1}\right)}$$