18.075 Practice Test I for In–class Exam # 1 Fall 2004

Justify your answers. Cross out what is not meant to be part of your final answer. Total number of points: 50

I. (5 pts) Show that for any complex numbers z_1 and z_2,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

II. (5 pts) Find all the possible values of

$$(1 - \sqrt{3}i)^{1/3}.$$

III. (Total 10 pts)

1. (3 pts) Can the function $v(x, y) = 4xy + y$ be the imaginary part of an analytic function? Explain.
2. (5 pts) Determine all the functions $u(x, y)$ such that $u(x, y) + iv(x, y)$ is analytic
3. (2 pts) Find the analytic function $f(z)$ explicitly in terms of z so that

$$f(z) = u(x, y) + iv(x, y).$$

IV. (5 pts) Compute the line integral

$$\int_C \frac{(z^3 - 2)}{z^4} dz$$

where C is the left half–circle joining $-2i$ and $2i$.

V. (Total 12 pts) Let

$$f(z) = \frac{1}{z^2 - 5z + 6}.$$

1. (3 pts) Write $f(z)$ as a sum of two fractions, i.e.,

$$f(z) = \frac{A}{z - z_1} + \frac{B}{z - z_2};$$

calculate the constants A and B. What are the points z_1 and z_2?

2. (5 pts) Explain whether it is possible to expand f(z) in Laurent (or Taylor) power series of:

(i) z, that converges in the region $0 \leq |z| < 3$?

(ii) $z + 1$, that converges in the region $2 < |z + 1| < 3$?

(iii) $z + 1$, that converges in the region $3 < |z + 1|$?

3. (4 pts) Write the Laurent series expansion of $f(z)$ in $|z-2| < 1$ as a power series of $(z-2)$.

VI. (5 pts) Let

$$f(z) \;=\; \frac{1}{(z^2+2)(z^2+3)}.$$

Compute the integral of $f(z)$ on the circles of center $-i$ and radii $1/4$, 1, and 4, respectively.

VII. (Total 5 pts) Show in an easy way that $\oint_C dz\, f(z) = 0$ where C is the circle of radius 1 centered at the origin, and

1. (1 pt) $f(z) = e^{z^2} \sin z$
2. (2 pts) $f(z) = \frac{1}{z^{10}}$
3. (2 pts) $f(z) = \tan z$

VIII. (3 pts) Prove the Cauchy Integral formula,

$$\oint_C \frac{f(\alpha)}{\alpha - b}\, d\alpha \;=\; 2\pi i f(b),$$

where C is a closed contour with the point b in its interior and $f(z)$ is a function analytic everywhere.