ex $x^3 y'' + y = 0$. try power series : $y = \sum_{n=0}^{\infty} a_n a^n \rightarrow \boxed{a_n = 0}$

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$

$x^3 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n+1} = \sum_{n=0}^{\infty} (n-1)(n-2) a_{n-1} x^n$

$a_{-1} = 0$

ODE: $\sum \{ a_n + (n-1)(n-2) a_{n-1} \} x^n = 0$

$a_n = -(n-1)(n-2) a_{n-1}$

$a_0 = 0$

$a_1 = 0$ everything is zero! $y = 0$

$a_2 = 0$ ∴ this method is useless

$y'' + A_1(x) y' + A_2(x) y = 0$

regular singular point at 0

ex 1: $A_1 = 0$, $A_2 = 1$ ex 2: $A_1 = \frac{1+x}{x}$, $A_2 = -\frac{1}{x^2}$ ex 3: $A_1 = 0$, $A_2 = \frac{1}{x^3}$ ← singularities

regular point: where A_1 and A_2 are both analytic.

regular singular point: point $x = x_0$ which is singular but such that $(x-x_0)A_1$ and $(x-x_0)^2 A_2$ are analytic.

At a regular point, power series work. If A_1 and A_2 are analytic at some point $x = x_0$, then, you can find two independent solutions by expanding $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$(x-x_0)^2 y'' + (x-x_0)\underbrace{[(x-x_0)A_1]}_{\text{analytic}} + \underbrace{[(x-x_0)^2 A_2]}_{\text{analytic}} y = 0$$

let $x_0 = 0$

$A_1 = \frac{1}{x}\frac{P}{R} \qquad A_2 = \frac{1}{x^2}\frac{Q}{R}$

Cannonical Form: $R(x)\frac{d^2y}{dx^2} + \frac{1}{x}P(x)\frac{dy}{dx} + \frac{1}{x^2}Q(x)\, y = 0$

$R(x) = 1 + R_1 x + R_2 x^2 + \dots$

$Q(x) = Q_0 + Q_1 x + Q_2 x^2 + \dots$

$P(x) = P_0 + P_1 x + P_2 x^2 + \dots$

$$\boxed{y = x^s \sum_{n=0}^{\infty} a_n x^n}$$

$x^2 y'' + xay' + by = 0$

$y = x^s$

$\underbrace{[s(s-1) + sa + b]}_{0} x^s = 0$

from previous page:

$\frac{dy}{dx} = a_0 s x^{s-1} + a_1(s+1)x^s + a_2(s+2)x^{s+1} + \ldots$

$y'' = a_0 s(s-1)x^{s-2} + a_1(s+1)s\,x^{s-1} + a_2(s+2)(s+1)x^3 + \ldots$

ODE: $(1 + R_1 x + R_2 x^2 + \ldots)[s(s-1)a_0 x^{s-2} + (s+1)s a_1 x^{s-1} + \ldots] + $ etc. (plug in everything)

they all start at the same power of x!

$[\underbrace{s(s-1) + P_0 s + Q_0}_{f(s)}]a_0 x^{s-2} + \ldots$

$\{\underbrace{[s(s-1)R_1 + sP_1 + Q_1]}_{g_1}a_0 + \overbrace{[s(s+1) + (s+1)P_0 + Q]}^{f(s+1)}a_1\} x^{s-1}$

$\{\underbrace{[s(s-1)R_2 + sP_2 + Q_2]}_{g_2}a_0 + \underbrace{[s(s+1)R_1 + (s+1)P_1 + Q_1]}_{g_1}a_1 + \overbrace{[(s+2)(s+1) + P_0(s+2) + Q_0]}^{f(s+2)}a_2\} x^s$

$f(s) = s(s-1) + P_0 s + Q_0$ $\quad g_n(s) = R_n(s-n)(s-n+1) + P_n(s-n) + Q_n$ $\quad f(s+n)a_n x^{s-2+n}$ term

$0 = Ly = f(s)a_0 x^{s-2} + [f(s+1)a_1 + g_1(s+1)a_0]x^{s-1} + \ldots + [f(s+n)a_n + \sum_{k=1}^{n} g_k(s+n)a_{n-k}]x^{s+n-2}$

Can take arbitary $a_0 \to a_0 = 1$

$a_1 = \frac{-g(s+1)a_0}{f(s+1)}$ works if $f(s+1) \neq 0$

$a_2 = \frac{1}{f(s+2)}\{g_1(s+2)a_1 + g_2(s+2)a_0\}$ works for $f(s+2) \neq 0$

$f(s+n) \neq 0$ for all $n \geq 1$

At a regular singular point, solve $f(s) = 0 \Rightarrow s = s_1$ and s_2.
<u>if</u> $s_2 - s_1$: not integer, 2 sol'ns $\quad s_2 - s_1$: integer, one solution.