Eigenvalues, eigenfunctions, orthogorality of eigenfunctions

$Mv = \lambda v$ λ = constant v = eigenvector, λ = eigenvalue

2×2 matrix $\rightarrow$ 2 eigenvectors v_1, v_2 $\vec{v}_1 = \vec{v}_2 = 0$

String:

$$\frac{d^2y}{dx^2} + k^2 y = 0, \quad \underbrace{\frac{d^2y}{dx^2}}_{M} = \underbrace{-k^2}_{\lambda} y \qquad y(0) = y(L) = 0$$

trivial solution = 0

non-trivial solution: $y(x) = A\cos kx + B\sin kx$

$y(0) = 0 \rightarrow \quad A = 0$

$y(L) = 0 \rightarrow \quad 0 = B\sin kl \qquad \sin kl = 0 \qquad kl = n\pi \qquad \lambda = k^2 = \left(\frac{n\pi}{l}\right)^2$

eigenfunction: $\sin\left(\frac{n\pi}{l}x\right)$ infinitlly many eigenvalues

orthogonal: $\vec{v}_1 \cdot \vec{v}_2 = 0 \qquad \vec{v}_1 = a\hat{i} + b\hat{j}, \quad \vec{v}_2 = c\hat{i} + d\hat{j}$

$\vec{v}_1 \cdot \vec{v}_2 = ac + bd = 0$

$$y_m\left[\frac{\partial^2 y_n}{\partial x^2} + k_n^2 y_n = 0\right]$$
$$-\ y_n\left[\frac{d^2 y_m}{dx} + k_m^2 y_m = 0\right]$$
$$\int_0^l \underbrace{y_m y''_n - y_n y''_m}_{0} + (k_n^2 - k_m^2) y_n y_m = 0$$

$$\int_0^l dx\, y_m y_n'' = \int_0^l y_m\, d(y'_n) \qquad dy'_n = y''_n dx \qquad \int u\,dv = uv - \int v\,du$$
$$= \cancel{y_m y'_n\Big|_0^l}^{\,0} - \int_0^l y'_m y'_n dx$$
$$= -\int_0^l y'_m y'_n dx$$
$$\int_0^l dx\, y_n y''_m = -\int_0^l y'_n y'_m dx$$

$$(k_n^2 - k_m^2)\int_0^l y_n y_m dx = 0 \qquad n \neq m$$

$$\int_0^l y_n y_m dx = 0 \qquad \therefore y_n \text{ and } y_m \text{ are orthogonal.}$$

$$\frac{d^2y}{dx^2} + k^2 y = 0 \qquad y(0)=0 \qquad y(\ell)=0$$
$$y'(\ell) + \sigma y(\ell) = 0$$

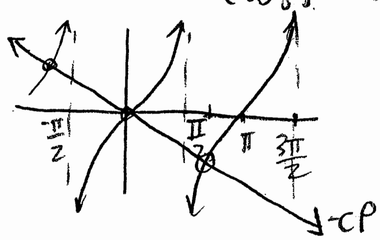
$y(x) = A\cos kx + B\sin kx$

$y(0)=0 \rightarrow A=0 \qquad y(x) = B\sin kx \qquad y' = kB\cos kx$

~~$y(\ell)=0 \rightarrow B\sin k\ell = 0 \quad k\ell = \pi n \quad k = \frac{\pi n}{\ell}$~~

$y'(\ell) + \sigma y(\ell) = 0 \rightarrow kB\cos k\ell + \sigma B\sin k\ell = 0$ let $k\ell = P$

$\frac{P}{\ell}\cos P + \sigma \sin P = 0 \qquad \tan P = -\frac{P}{\sigma \ell} = -cP$ let $c = (\sigma \ell)^{-1}$



$$\sin\alpha \sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$I_{nm} = \int_0^\ell y_n(x) y_m(x)\,dx = \int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi}{\ell}x\right)dx$$

$$= \frac{1}{2}\int_0^\ell \left\{ \cos\left[\frac{(n-m)\pi}{\ell}x\right] - \cos\left[\frac{(n+m)\pi}{\ell}\pi x\right]\right.$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{n-m}{\ell}\pi x\right)}{\frac{(n-m)\pi}{\ell}} - \text{preceding term } m \to -m\right]\Bigg|_0^\ell = 0$$

unless $n = m$

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

even function

$$I_{nn} = \int_0^\ell \frac{1-\cos\frac{2\pi n}{\ell}x}{2}\,dx = \frac{1}{2}\ell$$

$$\int_0^\ell y_n^2\,dx = \frac{\ell}{2} \qquad y_n = \sin\frac{n\pi}{\ell}x \qquad y_n = \sqrt{\frac{2}{\ell}}\sin\left(\frac{n\pi}{\ell}\right)x$$

$$\left.\begin{array}{l}\int_0^\ell y_n^2\,dx = 1 \\ \int_0^\ell y_n(x) y_m(x) = 0\end{array}\right\} \int_0^\ell y_n(x) y_m(x)\,dx = \delta_{nm}$$

Sturm-Liouville problem:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0$$

$$y(0) = y(\ell) = 0$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\ell}x\right) \qquad 0 < x < \ell$$

Fourier