18.075 An Overview of the Frobenius method

A. Classification of singular points of linear, 2nd-order ode's

Consider the linear, 2nd-order & homogeneous ordinary differential equation (ode)

$$y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 , \quad \text{where} \quad y'(x) \equiv \frac{dy}{dx} . \qquad (1)$$

(i) A point x_0 is called ordinary if both $a_1(z)$ and $a_2(z)$ are analytic at x_0.

(ii) A point x_0 is called a singular point of the ode if $a_1(z)$ or $a_2(z)$ is NOT analytic at x_0. In this case, x_0 is either

(a) a regular singular point, if

$$(z-x_0)a_1(z): \text{analytic at } x_0 \quad \text{and} \quad (z-x_0)^2 a_2(z): \text{analytic at } x_0 ,$$

or

(b) an irregular singular point, otherwise.

WARNING: To check the analyticity of a function at a point $z_0 (= x_0)$, say of the functions $(z-x_0)a_1(z)$ and $(z-x_0)^2 a_2(z)$, it is NOT sufficient to show that the limits $\lim_{z \to x_0} [(z-x_0)a_1(z)]$ and $\lim_{z\to x_0}[(z-x_0)^2 a_2(z)]$ exist and are finite. One HAS to show that $(z-x_0)a_1(z)$ and $(z-x_0)^2 a_2(z)$ have a TAYLOR SERIES EXPANSION around x_0.

Examples: 1) $x^2y'' + xy' + (x^2 - p^2)y = 0 , \quad p \neq 0.$

Divide both sides by x^2: $\quad y'' + \frac{1}{x}y' + (1 - \frac{p^2}{x^2})y = 0 \; ; \quad a_1(x) = \frac{1}{x} , \quad a_2(x) = 1 - \frac{p^2}{x^2} .$

Hence, $a_1(z)$ and $a_2(z)$ are analytic everywhere except at $x_0 = 0$.

Hence, $x_0=0$ is a singular point of this ode (the sole one).

$(z-x_0)\, a_1(z) = z a_1(z) = 1$, $(z-x_0)^2 a_2(z) = z^2 a_2(z) = z^2 - p^2$: both analytic at $x_0=0$.

Thus, $x_0=0$ is a regular singular point of this ode.

Example 2 : $x(x-1)y'' + (x^2-1)y' + (x+x^2)y = 0.$

Clearly, $a_1(x) = \dfrac{x^2-1}{x(x-1)} = \dfrac{x+1}{x}$, $a_2(x) = \dfrac{x+x^2}{x(x-1)} = \dfrac{x+1}{x-1}$.

$a_1(z)$ has a (simple) pole at $x_0=0$ while $a_2(z)$ has a simple pole at $x_0=1$.

Hence, the points $x_0=0,1$ are singular points of this ode.

$$(z-x_0)\, a_1(z) = \begin{cases} (x_0=0:) \quad z a_1(z) = z+1 & : \text{analytic at } x_0=0 \\ (x_0=1:) \quad (z-1)\, a_1(z) = \dfrac{z^2-1}{z} & : \text{analytic at } x_0=1 \end{cases},$$

$$(z-x_0)^2 a_2(z) = \begin{cases} (x_0=0:) \quad z^2 a_2(z) = z^2\,\dfrac{z+1}{z-1} & : \text{analytic at } x_0=0 \\ (x_0=1:) \quad (z-1)^2 a_2(z) = z^2-1 & : \text{analytic at } x_0=1 \end{cases}$$

Hence, the points $x_0=0,1$ are regular singular points of this ode.

Example 3 : $(1-\cos x)y'' + (\sin^2 x)\, y' + x y = 0$

$$a_1(x) = \frac{\sin^2 x}{1-\cos x}, \quad a_2(x) = \frac{x}{1-\cos x}.$$

Possible singular points: $\cos x = 1 \iff x_n = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

Let $w = z - x_n$: $a_1(z) = \dfrac{\sin^2 w}{1-\cos w} \simeq \dfrac{(w - w^3/3! + \ldots)^2}{1-(1-w^2/2! + \ldots)} = \dfrac{\cancel{w^2}(1-w^2/3! + \ldots)^2}{\cancel{w^2}(\frac{1}{2!} - w^2/4! + \ldots)}$ as $w \to 0$ $(z \to x_n)$

Hence, $a_1(z)$ has a Taylor expansion at $z=x_n \Rightarrow a_1(z)$: analytic at $z=x_n$.

$$a_2(z) = \frac{w+x_n}{w^2(1/2! - w^2/4! + \ldots)} = \left(\frac{1}{w} + \frac{x_n}{w^2}\right)\frac{1}{\frac{1}{2!} - w^2/4! + \ldots} \quad \text{as} \quad w \to 0$$

Notice that $a_2(z)$ has a double pole at $w=0$ if $\underset{(x_n \neq 0)}{n \neq 0}$, and a simple pole if $\underset{(x_n = 0)}{n=0}$.

Hence, x_n are singular points of this ode.

It is easy to check that $(z-x_n)a_1(z)$ and $(z-x_n)^2 a_2(z)$ are both analytic at x_n.

Thus, x_n are regular singular points of this ode.

<u>Example 4</u> : $\sqrt{x}\, y'' + (\ln x)\sqrt{x}\, y' + (\sin\sqrt{x})\, y = 0.$

$$a_1(x) = \ln x, \qquad a_2(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}.$$

- $a_1(z)$ has a branch point at $x_0 = 0$ because $\ln z$ is a multiple-valued function.
- $a_2(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}} = \dfrac{(\sqrt{z}) - (\sqrt{z})^3/3! + \dots}{\sqrt{z}} = \dfrac{\cancel{\sqrt{z}}}{\cancel{\sqrt{z}}}\left(1 - \dfrac{z}{3!} + \dfrac{z^2}{5!} - \dots\right)$: analytic at $x_0 = 0$.

Thus, $x_0 = 0$ is a singular point of the ode (because of $a_1(z)$).

Notice that $(z-x_0) = z a_1(z) = z \ln z$ canNOT be analytic at $x_0 = 0$.

Hence, $x_0 = 0$ is an <u>irregular</u> singular point of this ode.

$$\boxed{\textbf{B.}\ \text{General remarks on the ode} \quad y'' + a_1(x) y' + a_2(x) y = 0}$$

The following results (theorems) are known about the 2 independent solutions of this ode:

(i) If $x = x_0$ is an <u>ordinary</u> point, then both independent solutions are analytic at $x = x_0$ and can be expanded in a Taylor series around $x = x_0$:

$$y(x) = \sum_{k=0}^{\infty} A_k (x - x_0)^k$$

[Later, we choose <u>$x_0 = 0$</u> without loss of generality.

<u>**Detail** on convergence</u>: The radius of convergence of this series is at least as large as the distance to the nearest singularity of $a_1(z)$, $a_2(z)$ in the complex z plane ($x \to z$: complex).

(ii) If $x = x_0$ is a <u>regular</u> singular point, there is at least one solution of the Frobenius form

$$y(x) = (x-x_0)^s \sum_{k=0}^{\infty} A_k (x-x_0)^k \quad , \qquad s = \text{constant}.$$

The power s has to be determined and is called exponent of the ode.

(iii) If $x = x_0$ is an irregular singular point, then at least one solution of the ode does not have the Frobenius form.

C. The Frobenius method of power series

If $x_0 = 0$ is an ordinary point or a regular singular point, ode (1) can be put in the form

$$R(x)\, y'' + \frac{1}{x} P(x)\, y' + \frac{1}{x^2} Q(x)\, y = 0 \, , \qquad (2)$$

where P, Q, R are analytic at $x_0 = 0$ and $R(0) \neq 0$; without loss of generality, take $R(0) = 1$.

With these definitions, (2) is called a canonical form of (1)

General Remarks:

1. The solutions to (2) are saught in the Frobenius form

REMEMBER THIS!

$$y(x) = x^s \sum_{k=0}^{\infty} A_k x^k \, , \qquad A_0 \neq 0. \qquad (3)$$

2. The exponent s is determined as the solution to the quadratic equation

$$\boxed{f(s) = 0} \Rightarrow (s = s_1 \text{ or } s = s_2) \qquad s_{1,2} = \frac{1-P_0}{2} \pm \frac{1}{2}\sqrt{(1-P_0)^2 - 4Q_0} \, .$$

where $f(s) = s(s-1) + P_0 s + Q_0$, $P_0 \equiv P(0)$, $Q_0 \equiv Q(0)$.

3. The possible values of s determine the nature of solution, i.e., whether (3) suffices or needs to be modified in order to get both independent solutions.

Summary of results for s (under ③):

(I) If $s_1 = s_2$, only 1 solution is generated by the Frobenius form (3); let this solution be $y_1(x) = A_0\, u_1(x)$.

The 2nd solution is, in principle, saught in the form

$$y_2(x) = C_1\, u_1(x) \ln x + \sum_{k=0}^{\infty} B_k\, x^{k+s_2}, \qquad C_1 \neq 0,$$

same as s_1

where B_k need to be determined as functions of C_1.

[Alternatively, a 2nd solution can be saught by reduction of order, i.e., by setting $y_2(x) = g(x)\, u_1(x)$ and finding a 1st-order ode for $g(x)$.]

(II) If $s_1 \neq s_2$ we distinguish the following cases:

(a) If $s_1 \neq s_2$ and s_1, s_2 do NOT differ by an integer, then both independent solutions of the ode can be found in Frobenius form (3).

(b) If $s_1 = s_2 + m$, where m is a positive integer, then the above procedure of Frobenius (form (3)) generates 1 solution for $s = s_1$ (largest exponent). Sometimes, it is possible to generate both solutions by trying $s = s_2$ [see more detailed discussion in classnotes]. If a 2nd solution can NOT be generated in the form (3), then a 2nd solution is saught in the form

$$y_2(x) = C\, u_1(x) \ln x + \sum_{k=0}^{\infty} B_k\, x^{k+s_2},$$

where $y_1(x) = A_0\, u_1(x)$ is the Frobenius solution for $s = s_1$.

$$\boxed{R(x)=1} \quad \boxed{P(x)=1} \quad \boxed{Q(x)=x^2-p^2}$$

Example : Find 1 solution of $y'' + \underset{\substack{\uparrow \\ a_1(x)}}{\frac{1}{x}} y' + \underbrace{(1-p^2/x^2)}_{a_2(x)} y = 0$, $p \geq 0$ (Bessel eq.)

Clearly, $x_0 = 0$ is a regular singular point of this ode (above, also in canonical form).

Frobenius method : Seek a solution in form

$$y(x) = x^s \sum_{k=0}^{\infty} A_k x^k , \quad \underline{A_0 \neq 0.}$$

$$y' = \sum_{k=0}^{\infty} (k+s) A_k x^{k+s-1} , \qquad y'' = \sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^{k+s-2}.$$

Substitute in ode:

$$x^{s-2} \sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^k + x^{s-2} \sum_{k=0}^{\infty} (k+s) A_k x^k + \underbrace{\sum_{k=0}^{\infty} A_k x^{k+s}}_{\text{let } \ell = k+2} - p^2 \sum_{k=0}^{\infty} A_k x^{k+s-2} = 0.$$

$$\Downarrow$$

$$\sum_{\ell=2}^{\infty} A_{\ell-2} x^{\ell+s-2} , \quad \text{set } A_{-2} = A_{-1} = 0.$$

(factor out x^{s-2})

$$\Rightarrow x^{s-2} \left\{ \sum_{k=0}^{\infty} (k+s)(k+s-1) A_k x^k + \sum_{k=0}^{\infty} (k+s) A_k x^k + \sum_{k=0}^{\infty} A_{k-2} x^k - p^2 \sum_{k=0}^{\infty} A_k x^k \right\} = 0$$

$$\Rightarrow x^{s-2} \sum_{k=0}^{\infty} \left\{ \left[(k+s)^2 - p^2 \right] A_k + A_{k-2} \right\} x^k = 0$$

RECURRENCE FORMULA

$$\Rightarrow \boxed{\left[(k+s)^2 - p^2 \right] A_k + A_{k-2} = 0} , \quad k = 0, 1, 2, \ldots , \quad A_{-2} = A_{-1} = 0.$$

$\underline{k=0}$: $(s^2 - p^2) A_0 = 0 \overset{A_0 \neq 0}{\Longrightarrow} \boxed{s = \pm p}$; $s_1 = p$, $s_2 = -p$; $s_1 - s_2 = 2p$

$\{$from $\underbrace{s^2 - p^2 = 0}_{f(s)=0}$, since $A_0 \neq 0\}$

$\underline{k=1}$: $[(s+1)^2 - p^2] A_1 = 0$ $\quad$ etc.

Take $\underline{s = s_1 = p \geq 0}$, since we seek 1 solution of this ode.

$\underline{k=1}$: $(2p+1) A_1 = 0 \overset{p \geq 0}{\Longrightarrow} A_1 = 0$

$\underline{k=2}$: $\qquad A_2 = -\dfrac{A_0}{2(2+2p)}$

$\underline{k=3}$: $\qquad A_3 = 0$

$\underline{k=4}$: $\quad A_4 = -\dfrac{A_2}{4(4+2p)} = \dfrac{A_0}{2\cdot 4\ (2+2p)\ (4+2p)}$

$\underline{k=5}$: $\quad A_5 = 0$

$\underline{k=6}$: $\quad A_6 = \dfrac{-A_4}{6(6+2p)} = \cdots = \dfrac{-A_0}{2^6\ 1\cdot 2\cdot 3\ (1+p)(2+p)(3+p)}$ $\qquad \underline{\text{etc}}$

(Note the general pattern!)

So, we get

$$A_k = \frac{(-1)^{k/2}\ A_0}{2^k\left[1\cdot 2\cdot 3\cdots\left(\frac{k}{2}\right)\right](1+p)(2+p)\cdots\left(\frac{k}{2}+p\right)} \qquad \text{for } k\text{:even},$$

$$A_k = 0, \qquad \text{for } k\text{:odd}.$$

Let $\underline{k=2\ell}$:

$$A_k = A_{2\ell} = \frac{(-1)^{\ell}\ A_0}{2^{2\ell}\underbrace{(1\cdot 2\cdot 3\ldots\ell)}_{\ell!}(1+p)(2+p)\ldots(\ell+p)}, \qquad \ell=1,2,3,\ldots$$

Solution (for $s=s_1=p$): $\quad y_1(x) = A_0 u_1(x) = A_0 2^p \sum_{\ell=0}^{\infty} \dfrac{(-1)^{\ell}}{\ell!\ (1+p)(2+p)\ldots(\ell+p)}\left(\dfrac{x}{2}\right)^{2\ell+p}$

[This is proportional to the Bessel function $J_p(x)$; see classnotes.]

(b) <u>Find a 2nd independent solution to the given ode for $p=0$.</u>

The first attempt is to replace $s_1=p$ by $s_2=-p$ (smallest exponent) and check whether we can repeat the above procedure.

Everything <u>seems</u> to work, and we get the former formulas with $p\to -p$:

$$A_k = A_{2\ell} = \frac{(-1)^{\ell}\ A_0}{2^{2\ell}\ \ell!\ \underbrace{(1-p)(2-p)\ldots(\ell-p)}_{(p\to -p \text{ from before})}}, \qquad \ell=1,2,\ldots$$

<u>HOWEVER</u>, the denominator blows up when $p=$ integer >0.

Of course, we also <u>know</u> that the Frobenius method <u>fails</u> when $p=0$.

We determine a 2nd solution when $\underline{p=0}$. What we will do also works when $p = \text{integer} > 0$ but the algebra is very lengthy and not so instructive.

Try $$y_2(x) = C \ln x \cdot u_1(x) + \sum_{k=0}^{\infty} B_k \, x^{2k+s_2} \longleftarrow \boxed{s_2 = -p = 0 \text{ in this case}}, \quad \underline{\underline{C \neq 0}},$$

where $u_1(x)$ is the solution of part (a) for $s = s_1 = p$.

Then, (Add)

$$x^2 y_2(x) = C x^2 u_1(x) \ln x + \underbrace{\sum_{k=0}^{\infty} B_k \, x^{2k+2}}_{\Downarrow \; k+1=\ell},$$

$$\sum_{\ell=1}^{\infty} B_{\ell-1} \, x^{2\ell} = \sum_{k=0}^{\infty} B_{k-1} \, x^{2k}, \quad \underline{B_{-1} = 0}$$

$$x y_2'(x) = C x u_1'(x) \ln x + C u_1(x) + \sum_{k=0}^{\infty} 2k B_k \, x^{2k},$$

$$x^2 y_2''(x) = C x^2 u_1''(x) \ln x + 2 C x u_1'(x) - C u_1(x) + \sum_{k=0}^{\infty} 2k(2k-1) B_k \, x^{2k}.$$

The ode satisfied for $p=0$ is: $\quad x^2 y''(x) + x y'(x) + x^2 y(x) = 0$.

Add all series to get

$$C \underbrace{\left[x^2 u_1''(x) + \cancel{x u_1'(x)} + x^2 u_1(x) \right]}_{\text{0, because } u_1(x) \text{ is chosen to satisfy the ode!}} \ln x + 2 C x u_1'(x) + \sum_{k=0}^{\infty} \left[4k^2 B_k + B_{k-1} \right] x^{2k} = 0. \quad (4)$$

We now need to replace $u_1(x)$ by the series from part (a) for $\underline{p=0}$:

$$u_1(x) = \sum_{\ell=0}^{\infty} \bar{A}_\ell \, x^{2\ell}, \qquad \bar{A}_\ell \equiv \frac{(-1)^\ell}{(\ell!)^2} \, 2^{-2\ell}.$$

$$x u_1'(x) = \sum_{\ell=0}^{\infty} 2\ell \bar{A}_\ell \, x^{2\ell}.$$

Then (4) reads

$$\sum_{\ell=0}^{\infty} 4C\ell \bar{A}_\ell \, x^{2\ell} + \sum_{k=0}^{\infty} \left(4k^2 B_k + B_{k-1} \right) x^{2k} = 0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty} \left[4Ck\,\bar{A}_k + 4k^2 B_k + B_{k-1}\right] x^{2k} = 0$$

$$\Longleftrightarrow \underbrace{4Ck\,\bar{A}_k + 4k^2 B_k + B_{k-1} = 0}_{\text{RECURRENCE FORMULA FOR } B_k}, \quad k = 0,1,2,\ldots, \quad B_{-1} = 0.$$

$$\Longleftrightarrow 4Ck \frac{(-1)^k}{2^{2k}(k!)^2} + 4k^2 B_k + B_{k-1} = 0;$$ this satisfied for $k=0$. Take $\underline{k \neq 0}$.

To simplify the algebra define $\bar{B}_k$ such that

$$B_k = (-1)^{k+1} \frac{1}{2^{2k}(k!)^2} \bar{B}_k \Rightarrow B_{k-1} = (-1)^k \frac{4k^2}{2^{2k}(k!)^2} \bar{B}_{k-1}$$

In terms of $\bar{B}_k$, the recurrence relation is

$$4Ck \frac{(-1)^k}{\cancel{2^{2k}(k!)^2}} + 4k^2 (-1)^{k+1} \frac{1}{\cancel{2^{2k}(k!)^2}} \bar{B}_k + (-1)^k \frac{4k^2}{\cancel{2^{2k}(k!)^2}} \bar{B}_{k-1} = 0$$

$$\Longleftrightarrow \quad C = k(\bar{B}_k - \bar{B}_{k-1}) \Longleftrightarrow \frac{C}{k} = \bar{B}_k - \bar{B}_{k-1}; \qquad \underline{\underline{k \neq 0}}, \quad k = 1,2,3,\ldots$$

$$\left.\begin{array}{ll} \underline{k=1}: & C = \cancel{\bar{B}_1} - \bar{B}_0 \\ \underline{k=2}: & \frac{C}{2} = \cancel{\bar{B}_2} - \cancel{\bar{B}_1} \\ \vdots & \\ \underline{\underline{k=k}}: & \frac{C}{k} = \bar{B}_k - \cancel{\bar{B}_{k-1}} \end{array}\right\} \text{(Add all equations)}$$

$$\Rightarrow C\Big(\underbrace{1 + \frac{1}{2} + \cdots + \frac{1}{k}}_{\equiv \phi(k)}\Big) = \bar{B}_k - \bar{B}_0$$

It follows that $\bar{B}_k = C\,\phi(k) + \bar{B}_0$, where $\phi(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

Hence, $$B_k = \frac{(-1)^{k+1}}{2^{2k}(k!)^2}\left[C\,\phi(k) + B_0\right].$$

$$\Rightarrow y_2(x) = C \ln x\, u_1(x) + C \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2^{2k}(k!)^2} \phi(k)\, x^{2k} - B_0\, u_1(x)$$

$\{$arbitrary$\}$; can set $\boxed{B_0 = 0}$

In fact, this is the general solution to the given ode (for $\underline{p=0}$)! For $\underline{B_0 = 0}$ we get $Y_0(x)$